ON INFINITE GROUPS GENERATED BY TWO QUATERNIONS

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Abstract. Let x, y be two integral quaternions of norm p and l, respectively, where p, l are distinct odd prime numbers. We investigate the structure of $\langle x, y\rangle$, the multiplicative group generated by x and y. Under a certain condition which excludes $\langle x, y\rangle$ from being free or abelian, we show for example that $\langle x, y\rangle$, its center, commutator subgroup and abelianization are finitely presented infinite groups. We give many examples where our condition is satisfied and compute as an illustration a finite presentation of the group $\langle 1+j+k, 1+2j\rangle$ having these two generators and seven relations. In a second part, we study the basic question whether there exist commuting quaternions x and y for fixed p, l, using results on prime numbers of the form $r^2 + ms^2$ and a simple invariant for commutativity.

0. Introduction

Let p, l be two distinct odd prime numbers and x, y two integral Hamilton quaternions whose norms are in the set $\{p^r l^s : r, s \in \mathbb{N}_0\} \setminus \{1\}$. We are interested in the structure of the multiplicative group generated by x and y. These groups $\langle x, y\rangle$ are always infinite since x and y have infinite order by the assumption made on the norms.

We first consider the case where x and y do not commute. It is well-known that certain pairs x, y generate a free group of rank two, e.g. $\langle 1 + 2i, 1 + 2k\rangle \cong F_2$. However, if the norms of x and y are not powers of the same prime number p, then the structure of $\langle x, y\rangle$ is unknown in general. Nevertheless, we have shown in [11, Proposition 27] that the group $\langle 1 + 2i, 1 + 4k\rangle$ is *not* free by establishing an explicit relation of length 106. This holds in a more general situation. We will describe in Section 1 a homomorphism $\psi_{p,l}$ defined on the group of invertible rational quaternions, and give the definition of a group $\Gamma_{p,l}$ such that $\langle \psi_{p,l}(x), \psi_{p,l}(y)\rangle$ is a subgroup of $\Gamma_{p,l}$. It follows from results in [11] that if x, y satisfy some technical condition on the parity of their coefficients and if the group $\langle \psi_{p,l}(x), \psi_{p,l}(y)\rangle$ has finite index in $\Gamma_{p,l}$, then $\langle x, y\rangle$ is not free. Here we will use similar techniques to get more precise information on some groups $\langle x, y\rangle$ and naturally related subgroups or quotients. For instance, under the assumptions mentioned above which imply that $\langle x, y\rangle$ is not free, we will prove in Theorem 12 that $\langle x, y\rangle$ as well as its center, its commutator subgroup and its abelianization are finitely presented infinite groups. This contrasts to the case where $\langle x, y\rangle \cong F_2$, since then the center is finite (in fact trivial) and the commutator subgroup is not finitely presented (in fact not finitely generated). It also contrast to the case where x and y commute, since then the commutator subgroup of $\langle x, y\rangle$ is trivial. However, we will show in Theorem 12

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that our groups $\langle x, y\rangle$ contain non-abelian free subgroups F_2 of infinite index and free abelian subgroups $\mathbb{Z}\times\mathbb{Z}$ of infinite index.

Our constructions will be illustrated in Section 3 for the concrete example $x = 1 + j + k$ of norm $p = 3$ and $y = 1 + 2j$ of norm $l = 5$. In particular, we will compute a finite presentation of $\langle 1 + j + k, 1 + 2j\rangle$ having seven relations, and determine its center, which turns out to be $\langle 3^4, 5^4\rangle = \langle 81, 625\rangle < \mathbb{Q}^*$, a group isomorphic to $\mathbb{Z}\times\mathbb{Z}$. We guess that the finiteness assumption made on the index of $\langle \psi_{p,l}(x), \psi_{p,l}(y)\rangle$ in $\Gamma_{p,l}$ is a rather restrictive condition in general. Nevertheless we are able to give many explicit examples where it holds (at least for small p and l). Table 2 in Section 5 describes 191 selected examples of such pairs x, y of norm p and l, respectively, for 56 distinct pairs (p, l) satisfying $3 \le p < l < 100$.

If x and y commute, then the structure of $\langle x, y\rangle$ is less challenging. We get an abelian group like

$$\langle 1 + 2i, 1 + 4i\rangle \cong \mathbb{Z}\times\mathbb{Z}$$

or

$$\langle 1 + 2i, -1 - 2i\rangle = \langle 1 + 2i, -1\rangle \cong \mathbb{Z}\times\mathbb{Z}_2,$$

where we always use the notation $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for the cyclic group of order n. However, we are interested to know for which pairs p, l there exist commuting integral quaternions x, y of norm p and l at all. We will study this problem in Section 4 and give some partial general answers using congruence conditions for p and l.

1. Preliminaries

Throughout the whole article, let p, l be any pair of distinct odd prime numbers. The goal of this section is to define and describe the family of groups $\Gamma_{p,l}$ mentioned in the introduction. These groups are closely related to certain finitely generated multiplicative subgroups of invertible rational Hamilton quaternions.

For a commutative ring R with unit, let

$$\mathbb{H}(R) = \{x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in R\}$$

be the ring of Hamilton quaternions over R, i.e. $1, i, j, k$ is a free basis, and the multiplication is determined by the identities $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. Let $\overline{x} := x_0 - x_1 i - x_2 j - x_3 k$ be the *conjugate* of $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(R)$, and

$$|x|^2 := x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2 \in R$$

its *norm*. Note that $|xy|^2 = |x|^2|y|^2$ for all $x, y \in \mathbb{H}(R)$. We denote by $\Re(x) := x_0$ the "real part" of x. Let R^* be the multiplicative group of invertible elements in the ring R. We will mainly use the two groups $\mathbb{H}(\mathbb{Q})^* = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$ and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. Let

$$\mathbb{H}(R)_1 := \{x \in \mathbb{H}(R)^* : |x|^2 = 1\}$$

be the subgroup of quaternions of norm 1.

If K is a field, let as usual $\mathrm{PGL}_2(K) = \mathrm{GL}_2(K)/Z\mathrm{GL}_2(K)$ be the quotient of the group of invertible (2×2)-matrices over K by its center. We write brackets $[A] \in \mathrm{PGL}_2(K)$ to denote the image of the matrix $A \in \mathrm{GL}_2(K)$ under the quotient homomorphism $\mathrm{GL}_2(K) \to \mathrm{PGL}_2(K)$. Let $\mathbb{Q}_p$, $\mathbb{Q}_l$ be the field of p-adic and l-adic numbers, respectively, and fix elements $c_p, d_p \in \mathbb{Q}_p$ and $c_l, d_l \in \mathbb{Q}_l$ such that

$$c_p^2 + d_p^2 + 1 = 0 \in \mathbb{Q}_p \quad\text{and}\quad c_l^2 + d_l^2 + 1 = 0 \in \mathbb{Q}_l.$$

For $q \in \{p, l\}$ let ψ_q be the homomorphism of groups $\mathbb{H}(\mathbb{Q})^* \to \mathrm{PGL}_2(\mathbb{Q}_q)$ defined by

$$\psi_q(x) := \left[\begin{pmatrix} x_0 + x_1 c_q + x_3 d_q & -x_1 d_q + x_2 + x_3 c_q \\ -x_1 d_q - x_2 + x_3 c_q & x_0 - x_1 c_q - x_3 d_q \end{pmatrix}\right],$$

where $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Q})^*$. The following homomorphism $\psi_{p,l}$ will play a crucial role in our analysis of quaternion groups. Let

$$\psi_{p,l} : \mathbb{H}(\mathbb{Q})^* \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

be given by

$$\psi_{p,l}(x) := (\psi_p(x), \psi_l(x)).$$

This homomorphism is not injective, in fact (see [10, Chapter 3])

$$\ker(\psi_{p,l}) = Z(\mathbb{H}(\mathbb{Q})^*) = \{x \in \mathbb{H}(\mathbb{Q})^* : x = \overline{x}\} \cong \mathbb{Q}^*,$$

where for the last isomorphism we identify $\mathbb{Q}^*$ with the image of the natural injective homomorphism $\mathbb{Q}^* \to \mathbb{H}(\mathbb{Q})^*$ given by $x_0 \mapsto x_0 + 0 \cdot i + 0 \cdot j + 0 \cdot k$. In particular, we have $\psi_{p,l}(x) = \psi_{p,l}(y)$, if and only if $y = \lambda x$ for some $\lambda \in \mathbb{Q}^*$, and therefore

$$\psi_{p,l}(x) = \psi_{p,l}(-x)$$

for each $x \in \mathbb{H}(\mathbb{Q})^*$. Moreover, using the rule $\overline{x} = |x|^2 x^{-1}$, we also get

$$\psi_{p,l}(\overline{x}) = \psi_{p,l}(x)^{-1}.$$

For an odd prime number q, let X_q be the finite set of integral quaternions

$$\begin{aligned} X_q := \{x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})\,; \quad & |x|^2 = q\,; \\ & x_0 \text{ odd}, x_1, x_2, x_3 \text{ even}, \text{ if } q \equiv 1 \pmod 4\,; \\ & x_1 \text{ even}, x_0, x_2, x_3 \text{ odd}, \text{ if } q \equiv 3 \pmod 4\}\,. \end{aligned}$$

Observe that X_q has exactly $2(q+1)$ elements (by Jacobi's theorem on the number of representations of an integer as a sum of four squares) and that X_q is closed under conjugation and under multiplication by -1. As examples we have

$$X_3 = \{\pm 1 \pm j \pm k\},$$

where all of the 2^3 possible combinations of signs are allowed, and

$$X_5 = \{\pm 1 \pm 2i, \pm 1 \pm 2j, \pm 1 \pm 2k\}.$$

Finally, let $Q_{p,l}$ be the subgroup of $\mathbb{H}(\mathbb{Q})^*$ generated by $(X_p \cup X_l) \subset \mathbb{H}(\mathbb{Z})$ and let $\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$ be its image $\psi_{p,l}(Q_{p,l})$. Using the properties $\psi_{p,l}(x) = \psi_{p,l}(-x)$ and $\psi_{p,l}(\overline{x}) = \psi_{p,l}(x)^{-1}$ mentioned above, it follows that $\Gamma_{p,l}$ is generated by $(p+1)/2 + (l+1)/2$ elements. For example $\Gamma_{3,5}$ is generated by the five elements $\psi_{3,5}(1 + j + k)$, $\psi_{3,5}(1 + j - k)$, $\psi_{3,5}(1 + 2i)$, $\psi_{3,5}(1 + 2j)$ and $\psi_{3,5}(1 + 2k)$. See Section 3 for a finite presentation of $\Gamma_{3,5}$.

We recall some known properties of the group $\Gamma_{p,l}$ from [1], [8], [10], [11], [13]. It has a finite presentation with generators

$$a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}$$

and $(p+1)(l+1)/4$ defining relations of the form $ab\tilde{a}\tilde{b} = 1$ for some

$$a, \tilde{a} \in \{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} \text{ and } b, \tilde{b} \in \{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1}.$$

It acts freely and transitively on the vertices of the product of two regular trees of degree $p+1$ and $l+1$, respectively, is CAT(0), bi-automatic and can be decomposed as an amalgamated product of finitely generated free groups (see [1] or [10]). It is

CSA, i.e. all maximal abelian subgroups are malnormal (see [13, Proposition 2.6]), in particular it is commutative transitive, i.e. the relation of commutativity is transitive on non-trivial elements. It is also linear (see [11, Proposition 31] for an explicit injective homomorphism $\Gamma_{p,l} \to \mathrm{SO}_3(\mathbb{Q}) < \mathrm{GL}_3(\mathbb{Q})$), contains free abelian subgroups $\mathbb{Z}\times\mathbb{Z}$ (see [10, Proposition 4.2(3)]) as well as non-abelian free subgroups, for example

$$\langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle \cong F_{\frac{p+1}{2}} \text{ or } \langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle \cong F_{\frac{l+1}{2}}.$$

(However, it is not known whether there are elements

$$a \in \langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle \text{ and } b \in \langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle$$

generating a free group $\langle a, b \rangle \cong F_2$.) The conditions on the parity of x_0, x_1, x_2, x_3 in the definition of X_p and X_l are mainly used to guarantee that $\Gamma_{p,l}$ is torsion-free (see [8, Proposition 3.6] and [10, Theorem 3.30(4)]). Since finitely generated, torsion-free, virtually free groups are free (see [14]), the group $\Gamma_{p,l}$ is not virtually free. Moreover, $\Gamma_{p,l}$ is not virtually abelian. (It is well-known that the property of being virtually abelian is invariant under quasi-isometry for finitely generated groups. Now we use that $\Gamma_{p,l}$ is quasi-isometric to the non-virtually abelian group $F_2 \times F_2$, see [10, Proposition 4.25(4)]. Alternatively, without using finite index subgroups of $F_2 \times F_2$, we note that $\Gamma_{p,l}$ is also quasi-isometric to the non-abelian finitely presented torsion-free simple groups described in [1] and [10]. Infinite non-abelian simple groups are obviously not virtually abelian, which gives another proof that $\Gamma_{p,l}$ is not virtually abelian.) Any non-trivial normal subgroup of $\Gamma_{p,l}$ has finite index (by the "Normal Subgroup Theorem" of Burger-Mozes [1, Chapter 4 and 5]). This property also holds for any finite index subgroup of $\Gamma_{p,l}$.

2. The non-commutative case

We begin with some basic notations and a general lemma which will be applied to our quaternion groups later. If G is any group and $g_1, g_2 \in G$ two elements, we denote by $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ the commutator of g_1 and g_2, by G' the commutator subgroup of G, by G^{ab} the quotient G/G', and by ZG or $Z(G)$ the center of G.

Lemma 1. *Let G be a (multiplicatively written) group and N a normal subgroup of G. Then we have the following commutative diagram with exact rows and exact columns*

$$\begin{tikzcd}
 & 1 & 1 & 1 & \\
1 \arrow[r] & (G/N)' \arrow[u] \arrow[r, "i_1"] & G/N \arrow[u] \arrow[r, "p_1"] & (G/N)^{ab} \arrow[u] \arrow[r] & 1 \\
1 \arrow[r] & G' \arrow[u, "q_1"] \arrow[r, "i_2"] & G \arrow[u, "q_2"] \arrow[r, "p_2"] & G^{ab} \arrow[u, "q_3"] \arrow[r] & 1 \\
1 \arrow[r] & N \cap G' \arrow[u, "j_1"] \arrow[r, "i_3"] & N \arrow[u, "j_2"] \arrow[r, "p_3"] & p_2(N) \arrow[u, "j_3"] \arrow[r] & 1 \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

where i_1, i_2, i_3, j_1, j_2, j_3 are the natural homomorphisms injecting the corresponding normal subgroups, p_1, p_2, q_2 are the natural projections, p_3 is the restriction of p_2 to N, q_1 is induced by

$$q_1([g_1, g_2]) := [g_1N, g_2N] = [g_1, g_2]N,$$

i.e. $q_1(g') = g'N$, if $g' \in G'$, and finally $q_3(gG') := gN(G/N)'$.

Proof. It is clear that the top row, the middle row and the middle column are exact. The exactness of the bottom row and left column also follows immediately, since

$$\ker(p_3) = \{n \in N : nG' = G'\} = i_3(N \cap G')$$

and

$$\ker(q_1) = \{g' \in G' : g'N = N\} = j_1(N \cap G').$$

It remains to see that the right column is exact. The only non-obvious part is to show that $\ker(q_3) = \mathrm{im}(j_3)$. We have by definition of q_3

$$\ker(q_3) = \{gG' \in G/G' : gN \in (G/N)'\}.$$

Since

$$\begin{aligned} gN \in (G/N)' &\Leftrightarrow gN \in \{g'N \in G/N : g' \in G'\} \Leftrightarrow \exists\, g' \in G' : gN = g'N \\ &\Leftrightarrow \exists\, g' \in G' : g'^{-1}g \in N \Leftrightarrow \exists\, g' \in G', n \in N : g'^{-1}g = n \\ &\Leftrightarrow \exists\, g' \in G', n \in N : g'^{-1} = ng^{-1} \Leftrightarrow \exists\, n \in N : ng^{-1} \in G' \\ &\Leftrightarrow \exists\, n \in N : g^{-1}G' = n^{-1}G' \Leftrightarrow \exists\, n \in N : gG' = nG' \\ &\Leftrightarrow gG' \in \{nG' : n \in N\} \end{aligned}$$

it follows that

$$\ker(q_3) = \{nG' \in G/G' : n \in N\} = \mathrm{im}(j_3).$$

The commutativity of the diagram is a direct consequence of the given definitions. □

We refer to Section 1 for the notations concerning our quaternion groups. Let $x, y \in Q_{p,l}$ be two non-commuting quaternions and ψ the restriction of $\psi_{p,l}$ to the subgroup $\langle x, y\rangle < Q_{p,l} < \mathbb{H}(\mathbb{Q})^*$. We want to apply Lemma 1 to the situation where $q_2 : G \to G/N$ is the surjective homomorphism $\psi : \langle x, y\rangle \to \langle \psi(x), \psi(y)\rangle$ and where $N = \ker(\psi)$, in order to get some information on the structure of the four groups $\langle x, y\rangle$, $Z\langle x, y\rangle$, $\langle x, y\rangle'$ and $\langle x, y\rangle^{ab}$. First, we investigate the group $\ker(\psi)$ in Lemma 3(1),(2),(3), applying the following lemma. Then, we try to understand in Lemma 3(4) the bottom left term $N \cap G'$ of the diagram.

Lemma 2. (1) *The group $\mathbb{H}(\mathbb{Q})^*$ is commutative transitive on non-central elements, in other words $xz = zx$, $yz = zy$ implies $xy = yx$, whenever $x, y, z \in \mathbb{H}(\mathbb{Q})^* \setminus \mathbb{Q}^*$.*

(2) *The group $\mathbb{H}(\mathbb{Q})^*$ contains no subgroup isomorphic to a direct product of two non-abelian groups.*

Proof. (1) This is an elementary computation, see [10, Lemma 3.4(3)].

(2) Suppose that $\mathbb{H}(\mathbb{Q})^*$ contains a subgroup $G \times H$, where G, H are non-abelian groups. Take $g_1, g_2 \in G$ such that $g_1g_2 \neq g_2g_1$ and $h_1, h_2 \in H$ such that $h_1h_2 \neq h_2h_1$. Then clearly $g_1, g_2, h_1 \notin Z(\mathbb{H}(\mathbb{Q})^*)$. The fact that h_1 commutes with g_1 and with g_2 but $g_1g_2 \neq g_2g_1$ now contradicts part (1) of this lemma.

$\square$

Lemma 3. *Let $x, y \in Q_{p,l}$ be two non-commuting quaternions and ψ the restriction of $\psi_{p,l}$ to $\langle x, y\rangle$. Then*

1. $Z\langle x, y\rangle = \langle x, y\rangle \cap \mathbb{Q}^*$.
2. $\ker(\psi) = Z\langle x, y\rangle$.
3. $Z\langle x, y\rangle < \{\pm p^r l^s : r, s \in \mathbb{Z}\} = \langle -1, p, l\rangle \cong \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$, *in particular* $Z\langle x, y\rangle$ *is finitely presented.*
4. $Z\langle x, y\rangle \cap \langle x, y\rangle' < \{\pm 1\} \cong \mathbb{Z}_2$.
5. *The following diagram is commutative and has exact rows and columns*

$$\begin{tikzcd}
 & 1 & 1 & 1 & \\
1 \arrow[r] & \langle\psi(x),\psi(y)\rangle' \arrow[r] \arrow[u] & \langle\psi(x),\psi(y)\rangle \arrow[r] \arrow[u] & \langle\psi(x),\psi(y)\rangle^{ab} \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \langle x, y\rangle' \arrow[r] \arrow[u] & \langle x, y\rangle \arrow[r, "p_2"] \arrow[u, "\psi"] & \langle x, y\rangle^{ab} \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & 1 \text{ or } \{\pm 1\} \arrow[r] \arrow[u] & Z\langle x, y\rangle \arrow[r] \arrow[u] & p_2(Z\langle x, y\rangle) \arrow[r] \arrow[u] & 1 \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

   where the maps are defined as in Lemma 1, starting with

$$(N \longrightarrow G \xrightarrow{q_2} G/N) := (\ker(\psi) \longrightarrow \langle x, y\rangle \xrightarrow{\psi} \langle\psi(x), \psi(y)\rangle).$$

6. $[x, y] \neq -1$.
7. *Define the following three statements:*
   (i) $Z\langle x, y\rangle$ *is infinite.*
   (ii) $|x|^2 \neq 1$ *or* $|y|^2 \neq 1$.
   (iii) $\langle x, y\rangle^{ab}$ *is infinite.*
   Then (i) implies (ii) which in turn implies (iii).
8. *Let $r(\psi(x), \psi(y))$ be a relator in the group generated by $\psi(x)$, $\psi(y)$. Then* $r(x, y) \in Z\langle x, y\rangle$.
9. $\langle x, y\rangle \cong F_2$ *if and only if* $\langle\psi(x), \psi(y)\rangle \cong F_2$.

Proof. (1) First recall that $Z(\mathbb{H}(\mathbb{Q})^*) = \mathbb{Q}^*$. Since x, y do not commute by assumption, we have in particular $x, y \notin \mathbb{Q}^*$. If $z \in Z\langle x, y\rangle$, then $z \in \mathbb{Q}^*$ (otherwise x, y, z would be three non-central elements in $\mathbb{H}(\mathbb{Q})^*$ such that z commutes with x and y, but x and y do not commute, contradicting Lemma 2(1)). This shows that $Z\langle x, y\rangle < \langle x, y\rangle \cap \mathbb{Q}^*$.

To show $Z\langle x, y\rangle > \langle x, y\rangle \cap \mathbb{Q}^*$ we again use that $\mathbb{Q}^* = Z(\mathbb{H}(\mathbb{Q})^*)$.

(2) Using $\ker(\psi_{p,l}) = \mathbb{Q}^*$ and part (1) of this lemma, we get

$$\ker(\psi) = \ker(\psi_{p,l}) \cap \langle x, y\rangle = \mathbb{Q}^* \cap \langle x, y\rangle = Z\langle x, y\rangle.$$

(3) The norm of any element in $\langle x, y\rangle$ is of the form $p^r l^s$ for some $r, s \in \mathbb{Z}$, hence

$$Z\langle x, y\rangle = \langle x, y\rangle \cap \mathbb{Q}^* = \langle x, y\rangle \cap \{\pm p^r l^s : r, s \in \mathbb{Z}\} < \{\pm p^r l^s : r, s \in \mathbb{Z}\}.$$

The center $Z\langle x, y\rangle$ is finitely presented, since subgroups of finitely generated abelian groups are again finitely generated abelian, and since finitely generated abelian groups are finitely presented.

(4) As seen in part (1) of this lemma, we have $Z\langle x, y\rangle < \mathbb{Q}^*$. By the multiplicativity of the norm, any commutator in $\mathbb{H}(\mathbb{Q})^*$ has norm 1, hence any element in $\langle x, y\rangle'$ has norm 1. The only elements in $\mathbb{Q}^*$ having norm 1 are ± 1, and the statement follows.

(5) We combine Lemma 1 with part (2) and (4) of this lemma.

(6) $[x, y] = -1$ is equivalent to $xy = -yx$ which implies $\Re(x) = 0$ (see [10, Lemma 3.4(2)]). But then

$$x^2 = -x_1^2 - x_2^2 - x_3^2 \in \mathbb{Q}^*,$$

hence

$$1_{\Gamma_{p,l}} = \psi(x^2) = \psi(x)^2.$$

Since $\Gamma_{p,l}$ is torsion-free, $\psi(x) = 1_{\Gamma_{p,l}}$, and $x \in \ker(\psi) = \mathbb{Q}^* \cap \langle x, y\rangle$ by part (1) and (2) of this lemma, contradicting $\Re(x) = 0$.

(7) If $|x|^2 = |y|^2 = 1$, then all elements of $\langle x, y\rangle$ have norm 1, in particular all elements of $Z\langle x, y\rangle$ have norm 1. As in the proof of part (4) of this lemma we see that $Z\langle x, y\rangle < \{\pm 1\}$ is finite.

To show that (ii) implies (iii), we assume without loss of generality that $|x|^2 \neq 1$. Then the norms of x^m and x^n are distinct, whenever $m \neq n \in \mathbb{Z}$. Any element in $\langle x, y\rangle'$ has norm 1, as observed in the proof of part (4) of this lemma. It follows that any two cosets $x^m\langle x, y\rangle'$ and $x^n\langle x, y\rangle'$ are distinct. In particular, $\langle x, y\rangle'$ has infinite index in $\langle x, y\rangle$, hence $\langle x, y\rangle^{ab}$ is infinite.

(8) We have $r(x, y) \in \ker(\psi)$ and apply part (2) of this lemma.

(9) See [11, Proposition 32].

□

Remark 4. Going through the proofs of Lemma 3 we see that some results can be easily generalized: Let G be any non-abelian subgroup of $\mathbb{H}(\mathbb{Q})^*$. Then

- $\ker(\psi_{p,l}|_G) = ZG = G \cap \mathbb{Q}^*$.
- $ZG \cap G' < \{\pm 1\}$.
- ZG is infinite $\Rightarrow G \not< \mathbb{H}(\mathbb{Q})_1 \Rightarrow G^{ab}$ is infinite.

Related to Lemma 3(4), note that clearly $Z\langle x, y\rangle \cap \langle x, y\rangle' = 1$ if $x, y \in Q_{p,l}$ commute. We conjecture that $-1 \in Z\langle x, y\rangle \cap \langle x, y\rangle'$ is also impossible if $x, y \in Q_{p,l}$ do not commute (see Conjecture 7).

Question 5. *Let $x, y \in Q_{p,l}$ be two non-commuting quaternions. Is it possible that $-1 \in Z\langle x, y\rangle$? (Equivalently, is it possible that $-1 \in \langle x, y\rangle$?)*

Conjecture 6. *Let $x, y \in Q_{p,l}$. Then $-1 \notin \langle x, y\rangle'$. We even conjecture that $-1 \notin Q'_{p,l}$ (cf. Remark 15).*

As a consequence of Lemma 3(4) and Conjecture 6 we have

Conjecture 7. *Let $x, y \in Q_{p,l}$. Then $Z\langle x, y\rangle \cap \langle x, y\rangle' = 1$.*

In contrast to Lemma 3(6), commutators in $\mathbb{H}(\mathbb{Q})^*$ can of course be -1, for example $[i,j]=-1$. More generally, if $x,y\in\mathbb{H}(\mathbb{Q})^*$, then $[x,y]=-1$ if and only if $x_0=y_0=0$ and $x_1y_1+x_2y_2+x_3y_3=0$, see [12, Lemma 1]. In particular we have $-1\in(\mathbb{H}(\mathbb{Q})^*)'$ and $Z(\mathbb{H}(\mathbb{Q})^*)\cap(\mathbb{H}(\mathbb{Q})^*)'=\{\pm 1\}$. However, it is easy to proof that $-1\notin\langle X_q\rangle'$ for all odd prime numbers q.

Problem 8. *Characterize the (non-abelian) subgroups G of $\mathbb{H}(\mathbb{Q})^*$ such that $-1\in G'$.*

Before applying Lemma 3 to deduce our main results of this section in Theorem 12, we prove another general lemma and some statements about groups generated by quaternions $x,y\in Q_{p,l}$ satisfying a (relatively weak) norm condition in Proposition 10.

Lemma 9. *The only non-trivial element of finite order in $Q_{p,l}$ is -1. In particular, a subgroup $G<Q_{p,l}$ is torsion-free if and only if $-1\notin G$.*

Proof. Let $z\in Q_{p,l}$ such that $z^n=1$ for some $n\in\mathbb{N}$. Then $\psi(z)^n=1_{\Gamma_{p,l}}$. Since $\Gamma_{p,l}$ is torsion-free, we have $\psi(z)=1_{\Gamma_{p,l}}$, hence $z\in\mathbb{Q}^*$. But then $z\in\{\pm1\}$, since

$$1=|1|^2=|z^n|^2=(|z|^2)^n=z^{2n}.$$

□

Proposition 10. *Let $x,y\in Q_{p,l}$ be two quaternions of norms $|x|^2=p^{r_1}l^{s_1}$, $|y|^2=p^{r_2}l^{s_2}$, $r_1,r_2,l_1,l_2\in\mathbb{Z}$, such that $r_1s_2\neq r_2s_1$. (This condition holds for example if $|x|^2=p^r$, $|y|^2=l^s$ for some $r,s\in\mathbb{Z}\setminus\{0\}$). Then*

1. *$\langle x,y\rangle^{ab}\cong\mathbb{Z}\times\mathbb{Z}$, generated by the two commuting elements $x\langle x,y\rangle'$ and $y\langle x,y\rangle'$.*
2. *$\langle x,y\rangle\cap\mathbb{H}(\mathbb{Q})_1=\langle x,y\rangle'$.*
3. *$\langle x,y\rangle$ is torsion-free if and only if $\langle x,y\rangle'$ is torsion-free.*

Proof. (1) Let $r(x,y)=1$ be any relation in $\langle x,y\rangle$. Let s_x be the exponent sum of x in $r(x,y)$ and s_y the exponent sum of y in $r(x,y)$. Taking the norm of $r(x,y)=1$, we get

$$p^{s_xr_1+s_yr_2}\cdot l^{s_xs_1+s_ys_2}=1,$$

hence

$$\begin{pmatrix} r_1 & r_2\\ s_1 & s_2\end{pmatrix}\begin{pmatrix} s_x\\ s_y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

By assumption, the determinant $r_1s_2-r_2s_1$ is non-zero and we get $s_x=s_y=0$. This shows that the relator $r(x,y)$ is a consequence of the relator $[x,y]$. Thus $\langle x,y\rangle^{ab}\cong\langle x,y\mid[x,y]\rangle$, where the two generators correspond to $x\langle x,y\rangle'$ and $y\langle x,y\rangle'$. (Note that for any group G, a presentation of G^{ab} is obtained from a presentation $\langle X\mid R\rangle$ of G by adding to R all commutators of elements of X. In the case of a 2-generator group, we therefore have to add just a single commutator.)

(2) Clearly $\langle x,y\rangle\cap\mathbb{H}(\mathbb{Q})_1>\langle x,y\rangle'$.

To show the other inclusion, let $z\in\langle x,y\rangle\cap\mathbb{H}(\mathbb{Q})_1$. Since z has norm 1, it follows as in part (1) of this proposition that the exponent sums of x and y in z are 0. Thus $z\langle x,y\rangle'=\langle x,y\rangle'$ in the abelian group $\langle x,y\rangle/\langle x,y\rangle'$, in other words $z\in\langle x,y\rangle'$.

(3) Suppose that $\langle x, y\rangle$ is not torsion-free. Then $-1 \in \langle x, y\rangle$ by Lemma 9, hence $-1 \in \langle x, y\rangle'$ by part (2) of this proposition and $\langle x, y\rangle'$ is not torsion-free.
The other direction is obvious.

□

Lemma 11. *Let $x, y \in Q_{p,l}$ be two quaternions and assume that $\langle \psi(x), \psi(y)\rangle$ has finite index in $\Gamma_{p,l}$. Then*
(1) *x and y do not commute.*
(2) *$\langle \psi(x), \psi(y)\rangle^{ab}$ is finite.*

Proof. (1) If x and y commute, then $\langle \psi(x), \psi(y)\rangle$ is an abelian group, but $\Gamma_{p,l}$ is not virtually abelian.
(2) If G is a subgroup of finite index in $\Gamma_{p,l}$, then any non-trivial normal subgroup of G has finite index in G. Since $\langle \psi(x), \psi(y)\rangle$ is not abelian, the normal subgroup $\langle \psi(x), \psi(y)\rangle'$ is not trivial, hence has finite index in $\langle \psi(x), \psi(y)\rangle$.

□

Theorem 12. *Let $x, y \in Q_{p,l}$ be two quaternions and let ψ be the restriction of $\psi_{p,l}$ to $\langle x, y\rangle$. Assume that the group $\psi(\langle x, y\rangle) = \langle \psi(x), \psi(y)\rangle$ has finite index in $\Gamma_{p,l}$. Then*
(1) *The group $\langle x, y\rangle$ is finitely presented and infinite.*
(2) *The group $\langle x, y\rangle'$ is finitely presented and infinite.*
(3) *The following three statements are equivalent:*
(i) $Z\langle x, y\rangle$ is infinite.
(ii) $|x|^2 \neq 1$ or $|y|^2 \neq 1$.
(iii) $\langle x, y\rangle^{ab}$ is infinite.
(4) *The group $\langle x, y\rangle$ is not virtually solvable.*
(5) *The group $\langle x, y\rangle$ contains a free subgroup F_2 of infinite index.*
(6) *The group $\langle x, y\rangle$ contains a subgroup $\mathbb{Z} \times \mathbb{Z}$ of infinite index. In particular, $\langle x, y\rangle$ is not hyperbolic.*
(7) *Let N be a normal subgroup of $\langle x, y\rangle$ of infinite index such that $Z\langle x, y\rangle < N$. Then $N = Z\langle x, y\rangle$.*

Proof. By Lemma 11(1) we can use the commutative diagram of Lemma 3(5) which we will simply call "the diagram" in this proof.
(1) The group $\langle \psi(x), \psi(y)\rangle$ is finitely presented, since it has finite index in the finitely presented group $\Gamma_{p,l}$ by assumption. Using the middle column of the diagram, the group $\langle x, y\rangle$ is an extension of the finitely presented group $Z\langle x, y\rangle$ by the finitely presented group $\langle \psi(x), \psi(y)\rangle$, hence finitely presented.
It is clear that $\langle x, y\rangle$ is infinite, since $\langle \psi(x), \psi(y)\rangle$ is infinite as a finite index subgroup of the infinite group $\Gamma_{p,l}$.
(2) By Lemma 11(2) $\langle \psi(x), \psi(y)\rangle^{ab}$ is finite. By the exactness of the top row in the diagram, $\langle \psi(x), \psi(y)\rangle'$ has finite index in $\langle \psi(x), \psi(y)\rangle$, hence is also finitely presented. Now, using the exactness of the left column in the diagram, $\langle x, y\rangle'$ is an extension of a finite group ($\mathbb{Z}_2$ or 1) by the finitely presented group $\langle \psi(x), \psi(y)\rangle'$ and therefore finitely presented.
By the exactness of the top row and left column, $\langle \psi(x), \psi(y)\rangle'$ and $\langle x, y\rangle'$ are infinite groups.

(3) Because of Lemma 11(1) and Lemma 3(7), it remains to prove that (iii) implies (i). Therefore suppose that $\langle x, y\rangle^{ab}$ is infinite. Since $\langle \psi(x), \psi(y)\rangle^{ab}$ is finite by Lemma 11(2), the exactness of the right column in the diagram shows that $p_2(Z\langle x, y\rangle)$ is infinite. The exactness of the bottom row in the diagram now shows that $Z\langle x, y\rangle$ is infinite.

(4) We first show that the group $\langle \psi(x), \psi(y)\rangle$ is not virtually solvable. Note that the property of being virtually solvable is *not* invariant under quasi-isometry for finitely generated groups (see [5]), why we cannot use the same strategy as for the proof that $\Gamma_{p,l}$ is not virtually abelian. Instead of that, we adapt an idea already used in the proof of [13, Corollary 2.8]. Let $V < \langle \psi(x), \psi(y)\rangle$ be a subgroup of finite index. Then V is not abelian (otherwise $\Gamma_{p,l}$ would be virtually abelian). The group $\langle \psi(x), \psi(y)\rangle$ is a non-abelian CSA-group, since $\Gamma_{p,l}$ is CSA, and subgroups of CSA-groups are CSA ([9, Proposition 10(3)]). Now, we use the fact that non-abelian CSA-groups do not have any non-abelian solvable subgroups ([9, Proposition 9(5)]) to conclude that V is not solvable.

Let U be any finite index subgroup of $\langle x, y\rangle$. Since quotients of solvable groups are solvable, and

$$[\langle \psi(x), \psi(y)\rangle : \psi(U)] \leq [\langle x, y\rangle : U] < \infty,$$

the group U is not solvable, and therefore $\langle x, y\rangle$ is not virtually solvable.

(5) There is a well-known injective homomorphism of groups

$$\mathbb{H}(\mathbb{Q})^* \to \mathrm{GL}_4(\mathbb{Q})$$

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix},$$

in particular $\langle x, y\rangle$ is a finitely generated linear group in characteristic 0. Since $\langle x, y\rangle$ is not virtually solvable by part (4) of this theorem, it contains by the Tits Alternative ([15]) a free subgroup F_2.

A free subgroup F_2 of $\langle x, y\rangle$ always has infinite index, since otherwise $\psi(F_2)$ has finite index in $\Gamma_{p,l}$, but $\psi(F_2) \cong F_2$ by Lemma 3(9) and $\Gamma_{p,l}$ is not virtually free.

(6) Let $a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}$ be the standard generators of $\Gamma_{p,l}$. By [10, Proposition 4.2(3)], there are elements

$$a \in \langle a_1, \dots, a_{\frac{p+1}{2}}\rangle \text{ and } b \in \langle b_1, \dots, b_{\frac{l+1}{2}}\rangle,$$

such that a, b generate a subgroup of $\Gamma_{p,l}$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Since $\langle \psi(x), \psi(y)\rangle$ has finite index in $\Gamma_{p,l}$ by assumption, there are $m, n \in \mathbb{N}$ such that $a^m, b^n \in \langle \psi(x), \psi(y)\rangle$ and

$$\mathbb{Z} \times \mathbb{Z} \cong \langle a^m, b^n\rangle < \langle \psi(x), \psi(y)\rangle.$$

Let w_1, w_2 be quaternions in $\langle x, y\rangle$ such that $a^m = \psi(w_1)$ and $b^n = \psi(w_2)$. Since a^m commutes with b^n, we have

$$\psi(w_1 w_2) = \psi(w_2 w_1),$$

hence either

$$w_1 w_2 = -w_2 w_1$$

or

$$w_1w_2 = w_2w_1.$$

The first case can be excluded similarly as in Lemma 3(6). The commuting quaternions w_1, w_2 have infinite order, since $w_1, w_2 \notin \{-1, 1\}$. Moreover, it is not possible that $w_1^r = w_2^s$ for some $r, s \in \mathbb{N}$, since otherwise $\psi(w_1)^r = \psi(w_2)^s$, i.e. $a^{mr} = b^{ns}$. But since $mr \neq 0 \neq ns$, this is impossible using the "normal form" (see [10, Proposition 1.10]) for elements of $\Gamma_{p,l}$. Therefore

$$\mathbb{Z} \times \mathbb{Z} \cong \langle w_1, w_2 \rangle < \langle x, y \rangle.$$

This subgroup has to be of infinite index, since $\Gamma_{p,l}$ is not virtually abelian.

(7) The assumptions imply that $N/Z\langle x, y\rangle$ is a normal subgroup of $\langle x, y\rangle/Z\langle x, y\rangle$ of infinite index, since

$$(\langle x, y\rangle/Z\langle x, y\rangle)/(N/Z\langle x, y\rangle) \cong \langle x, y\rangle/N.$$

The group

$$\langle x, y\rangle/Z\langle x, y\rangle \cong \langle \psi(x), \psi(y)\rangle$$

has no non-trivial normal subgroups of infinite index by Lemma 11(2), hence $N/Z\langle x, y\rangle = 1$.

□

3. The example $\langle 1 + j + k, 1 + 2j\rangle$

We study in this section the group $\langle 1 + j + k, 1 + 2j\rangle$. Let $p = |1 + j + k|^2 = 3$ and $l = |1 + 2j|^2 = 5$. Then the group $\Gamma := \Gamma_{3,5}$ has the finite presentation

$$\Gamma = \langle a_1, a_2, b_1, b_2, b_3 \mid a_1b_1a_2b_2,\ a_1b_2a_2b_1^{-1},\ a_1b_3a_2^{-1}b_1, \\ a_1b_3^{-1}a_1b_2^{-1},\ a_1b_1^{-1}a_2^{-1}b_3,\ a_2b_3a_2b_2^{-1}\rangle,$$

where we take

$$\begin{aligned}
a_1 &:= \psi_{3,5}(1 + j + k), & a_1^{-1} &= \psi_{3,5}(1 - j - k),\\
a_2 &:= \psi_{3,5}(1 + j - k), & a_2^{-1} &= \psi_{3,5}(1 - j + k),\\
b_1 &:= \psi_{3,5}(1 + 2i), & b_1^{-1} &= \psi_{3,5}(1 - 2i),\\
b_2 &:= \psi_{3,5}(1 + 2j), & b_2^{-1} &= \psi_{3,5}(1 - 2j),\\
b_3 &:= \psi_{3,5}(1 + 2k), & b_3^{-1} &= \psi_{3,5}(1 - 2k).
\end{aligned}$$

In the following, let $x := 1 + j + k$, $y := 1 + 2j$, $a := a_1 = \psi_{3,5}(x)$, $b := b_2 = \psi_{3,5}(y)$ and define the five words

$$\begin{aligned}
r_1(a, b) &:= ba^2bab^{-1}a^4b^{-1}a,\\
r_2(a, b) &:= a^{-1}ba^{-1}ba^2ba^{-2}ba^{-1}b^2a^2bab,\\
r_3(a, b) &:= baba^2b^2ab^{-1}ab^2a^2bab^2,\\
r_4(a, b) &:= ba^2ba^{-1}b^{-3}a^{-2}b^{-1}ab^2,\\
r_5(a, b) &:= ab^{-1}a^2b^{-1}ab^{-1}a^{-2}b^{-1}a^{-2}ba^{-2}ba^{-1}ba^2ba
\end{aligned}$$

of lengths 12, 18, 18, 14 and 22, respectively. We will simply write $r_1, \ldots, r_5$ instead of $r_1(a, b), \ldots, r_5(a, b)$ or $r_1(x, y), \ldots, r_5(x, y)$ if the context is unambiguous, like in the presentations of $\langle a, b\rangle$ and $\langle x, y\rangle$ given below. Using GAP ([6]), we have done the following computations.

Observation 13. Let $\Gamma, a, a_2, b_1, b, b_3, x, y, r_1, r_2, r_3, r_4, r_5$ be as above. Then

(1) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.
(2) $(\Gamma')^{ab} \cong \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_{16}$.
(3) $[\Gamma : \langle a, b\rangle] = 2$ such that $a_2, b_1 \notin \langle a, b\rangle$ and $b_3 = ab^{-1}a$.
(4) $\langle a, b\rangle$ has the finite presentation $\langle a, b \mid r_1, r_2, r_3, r_4, r_5\rangle$. There is no shorter non-trivial freely reduced relator than r_1.
(5) $\langle a, b\rangle^{ab} \cong \langle a, b \mid r_1, r_2, r_3, r_4, r_5, [a, b]\rangle \cong \langle a, b \mid a^8, b^8, [a, b]\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_8$.
(6) $\langle a, b\rangle' = \langle [a, b], [a, b^{-1}], [a^{-1}, b], [a, b^2], [a^2, b]\rangle$.
(7) $(\langle a, b\rangle')^{ab} \cong \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_{64}$.
(8) $r_1(x, y) = 3^4$, $r_2(x, y) = 5^4$, $r_3(x, y) = 3^4 5^4$, $r_4(x, y) = r_5(x, y) = 1$.

By Lemma 3(3),(8) we have

$$\mathbb{Z} \times \mathbb{Z} \cong \langle 3^4, 5^4\rangle < Z\langle x, y\rangle < \langle -1, 3, 5\rangle \cong \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}.$$

It will turn out that

$$Z\langle x, y\rangle = \langle 3^4, 5^4\rangle = \langle r_1(x, y), r_2(x, y)\rangle.$$

This enables us to compute an explicit presentation of the group $\langle x, y\rangle$ in the following proposition:

Proposition 14. *Let* $\Gamma, a_1, a_2, b_1, b_2, b_3, x, y, a, b, r_1, r_2, r_3, r_4, r_5$ *be as above. Then*

(1) $Z\langle x, y\rangle = \langle 3^4, 5^4\rangle$.
(2) $\langle x, y\rangle$ *has a finite presentation*

$$\langle x, y \mid r_4,\ r_5,\ r_1 r_2 r_3^{-1},\ [x, r_1],\ [x, r_2],\ [y, r_1],\ [y, r_2]\rangle$$

$$\begin{aligned} = \langle x, y \mid \\ & yx^2yXY^3X^2Yxy^2, \\ & xYx^2YxYX^2YX^2yX^2yXyx^2yx, \\ & yx^2yxYx^3yx^2yX^2yXy^2x^2yxYXYX^2Y^2XyXY^2X^2YXY, \\ & xyx^2yxYx^4YXyX^4yXYX^2Y, \\ & yXyx^2yX^2yXy^2x^2yxyXYXYX^2Y^2xYx^2YX^2YxYx, \\ & y^2x^2yxYx^4YxYXyX^4yXYX^2Y, \\ & yXyXyx^2yX^2yXy^2x^2yxYXYX^2Y^2xYx^2YX^2YxYx\rangle, \end{aligned}$$

where we write $X := x^{-1}$ *and* $Y := y^{-1}$. *There is no shorter non-trivial freely reduced relator than* $r_4(x, y)$ *in* $\langle x, y\rangle$.

(3) $\langle x, y\rangle$ *is torsion-free.*
(4) $\langle x, y\rangle^{ab} \cong \mathbb{Z} \times \mathbb{Z}$ *and* $(\langle x, y\rangle')^{ab} \cong \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_{64}$.
(5) $\langle x, y\rangle'$ *has amalgam decompositions*

$$F_{65} *_{F_{385}} F_{65} \ \text{ and } \ F_{129} *_{F_{513}} F_{129}.$$

Moreover, the group $\langle x, y\rangle''$ *has amalgam decompositions*

$$F_{262145} *_{F_{1572865}} F_{262145} \ \text{ and } \ F_{524289} *_{F_{2097153}} F_{524289}.$$

(6) *The two quaternions*

$$y = 1 + 2j \ \text{ and } \ xy^{-1}x = 3 \cdot 5^{-1}(1 + 2k)$$

generate a free subgroup F_2 *of* $\langle x, y\rangle$.

(7a) *Let* $r := [x, yx^{-1}y]$ *and* $q := x^{-1}rx$. *Then the two quaternions*

$$r^2qr^4 = 3^{-2}5^{-12}(1700294841 + 519258632i - 556215472j + 1165319056k)$$

and

$$r^4qr^2 = 3^{-2}5^{-12}(1700294841 + 1191258632i + 283784528j + 661319056k)$$

generate a free subgroup F_2 *of* $\langle x, y\rangle'$.

(7b) *The two quaternions*

$$[xy^{-1}x, y] = 5^{-2}(-7 - 8i - 16j + 16k)$$

and

$$[y^{-1}, xy^{-1}x] = 5^{-2}(-7 - 8i - 16j - 16k)$$

generate a free subgroup F_2 *of* $\langle x, y\rangle'$.

(7c) *The two quaternions*

$$\frac{y^8}{5^4} = 5^{-4}(-527 + 336j)$$

and

$$\frac{5^4(xy^{-1}x)^8}{3^8} = 5^{-4}(-527 + 336k)$$

generate a free subgroup F_2 *of* $\langle x, y\rangle'$.

(8) *Let*

$$w_1 := xy^{-1}x^{-1}y^{-1}x^{-2}y^{-2} = 3^{-3}5^{-2}(5 + 4i + 6j - 2k)$$

and

$$w_2 := y^{-1}x^2y^{-1}xy^{-1}xy^{-1}x^2 = -3^45^{-4}(11/3 + 4i + 6j - 2k).$$

Then $\langle w_1, w_2\rangle$ *is a subgroup of* $\langle x, y\rangle$ *isomorphic to* $\mathbb{Z} \times \mathbb{Z}$.

Proof. (1) Let $z := 1 + j - k$, $s := 1 + 2i$, $t := 1 + 2k$, and let

$$G := \langle x, z, s, y, t\rangle < Q_{3,5}.$$

Our first goal is to obtain an explicit finite presentation of G. This will be useful to compute $Z\langle x, y\rangle$.

Let $u_1, \ldots, u_6$ be the six defining relators of Γ from above, i.e.

$$\begin{aligned}
u_1(a_1, a_2, b_1, b_2, b_3) &:= a_1b_1a_2b_2,\\
u_2(a_1, a_2, b_1, b_2, b_3) &:= a_1b_2a_2b_1^{-1},\\
u_3(a_1, a_2, b_1, b_2, b_3) &:= a_1b_3a_2^{-1}b_1,\\
u_4(a_1, a_2, b_1, b_2, b_3) &:= a_1b_3^{-1}a_1b_2^{-1},\\
u_5(a_1, a_2, b_1, b_2, b_3) &:= a_1b_1^{-1}a_2^{-1}b_3,\\
u_6(a_1, a_2, b_1, b_2, b_3) &:= a_2b_3a_2b_2^{-1}.
\end{aligned}$$

As in Lemma 3(3), it is easy to see that $ZG < \langle -1, 3, 5\rangle$. On the other hand, we check that

$$\begin{aligned}
-1 &= u_5(x, z, s, y, t) = xs^{-1}z^{-1}t \in G,\\
3 &= u_6(x, z, s, y, t) = ztzy^{-1} \in G,\\
5 &= -u_3(x, z, s, y, t) = -xtz^{-1}s \in G.
\end{aligned}$$

This shows that $ZG = \langle -1, 3, 5\rangle$. We note that this implies $G = Q_{3,5}$ and that G is not torsion-free.

Let ψ be the restriction of $\psi_{3,5}$ to G. As in Lemma 3(2), we have

$$\ker(\psi : G \twoheadrightarrow \Gamma) = ZG.$$

Knowing finite presentations of Γ and ZG, it is now possible to compute a finite presentation of the extension G (for example see [7, Chapter 10.2] for the detailed explicit general construction of such a presentation). Evaluating the three other defining relators of Γ

$$\begin{aligned} u_1(x, z, s, y, t) &= xszy = -15, \\ u_2(x, z, s, y, t) &= xyzs^{-1} = -3, \\ u_4(x, z, s, y, t) &= xt^{-1}xy^{-1} = 3/5, \end{aligned}$$

we get the finite presentation

$$\begin{aligned} G = \langle x, z, s, y, t \mid\ & [x, u_3],\ [x, u_5],\ [x, u_6], \\ & [z, u_3],\ [z, u_5],\ [z, u_6], \\ & [s, u_3],\ [s, u_5],\ [s, u_6], \\ & [y, u_3],\ [y, u_5],\ [y, u_6], \\ & [t, u_3],\ [t, u_5],\ [t, u_6], \\ & u_5^2,\ u_1 u_3^{-1} u_6^{-1},\ u_2 u_5 u_6^{-1},\ u_3 u_4 u_5 u_6^{-1}\rangle. \end{aligned}$$

We have seen that $\langle 3^4, 5^4\rangle < Z\langle x, y\rangle < \langle -1, 3, 5\rangle$. The group $\langle 3^4, 5^4\rangle$ has index $32 = 2 \cdot 4 \cdot 4$ in $\langle -1, 3, 5\rangle$, and we write $\langle -1, 3, 5\rangle$ as a finite disjoint union of cosets

$$\langle -1, 3, 5\rangle = \langle 3^4, 5^4\rangle \sqcup \lambda_2\langle 3^4, 5^4\rangle \sqcup \ldots \sqcup \lambda_{32}\langle 3^4, 5^4\rangle,$$

such that $\lambda_2, \ldots, \lambda_{32} \in \langle -1, 3, 5\rangle \setminus \langle 3^4, 5^4\rangle$. To prove our statement $Z\langle x, y\rangle = \langle 3^4, 5^4\rangle$, it is enough to check that the index condition

$$[G : \langle x, y\rangle] > [G : \langle x, y, \lambda_m\rangle]$$

is satisfied for all $m \in \{2, \ldots, 32\}$. To see why this is enough, assume that this index condition holds and that $Z\langle x, y\rangle \neq \langle 3^4, 5^4\rangle$. Then there is an element $\gamma \in Z\langle x, y\rangle \setminus \langle 3^4, 5^4\rangle$, hence $\gamma \in \lambda_m\langle 3^4, 5^4\rangle$ for some $m \in \{2, \ldots, 32\}$, and therefore $\lambda_m \in Z\langle x, y\rangle < \langle x, y\rangle$, which is impossible, since the index condition implies that $\lambda_m \notin \langle x, y\rangle$ for all $m \in \{2, \ldots, 32\}$.

Using GAP ([6]) and the finite presentation of G from above, we have checked that the condition $[G : \langle x, y\rangle] > [G : \langle x, y, \lambda_m\rangle]$ indeed holds. In particular, we have $[G : \langle x, y\rangle] = 64$ and $[G : \langle x, y, \lambda\rangle] = 32 < 64$, if

$$\lambda \in \{-1, 3^2, -3^2, 5^2, -5^2, 3^2 5^2, -3^2 5^2\}.$$

(2) We obtain the stated finite presentation of $\langle x, y\rangle$ as an extension of $Z\langle x, y\rangle$ by $\langle a, b\rangle$. Observe that the four commutator relators in the presentation express the fact that $r_1 = 3^4$ and $r_2 = 5^4$ are in the center of $\langle x, y\rangle$.

We have checked with GAP ([6]) that there is no non-trivial freely reduced relator of length less than 14. Moreover, any freely reduced relator of length 14 is conjugate to r_4 or to r_4^{-1}.

(3) The group $\langle x, y\rangle$ is torsion-free by Lemma 9, using that $-1 \notin Z\langle x, y\rangle$.

(4) The abelianization of $\langle x, y\rangle$ is easy to compute, once the finite presentation of $\langle x, y\rangle$ given in part (2) of this proposition is known. Alternatively, we apply Proposition 10(1).

Since $-1 \notin Z\langle x, y\rangle$, we have

$$\langle x, y\rangle' \cap Z\langle x, y\rangle = 1,$$

hence an isomorphism $\langle a, b\rangle' \cong \langle x, y\rangle'$ by the left column of our diagram of Lemma 3(5). Now, we use Observation 13(7).

(5) Since $\langle x, y\rangle' \cong \langle a, b\rangle'$, it is enough to show that $\langle a, b\rangle'$ has the claimed amalgam decompositions. Let Γ_0 be the kernel of the surjective homomorphism $\Gamma \to \mathbb{Z}_2 \times \mathbb{Z}_2$, given by

$$a_1, a_2 \mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \text{ and } b_1, b_2, b_3 \mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}).$$

Then Γ_0 has by [10, Proposition 1.4] amalgam decompositions $F_3 *_{F_{13}} F_3$ and $F_5 *_{F_{17}} F_5$. Commutators in Γ are obviously in Γ_0, in particular $\langle a, b\rangle' < \Gamma_0$. We have checked with GAP ([6]) that $\langle a, b\rangle' = \langle\!\langle [a, b]\rangle\!\rangle_{\langle a,b\rangle}$ is in fact $\langle\!\langle [a, b]\rangle\!\rangle_\Gamma$, the normal closure of $[a, b]$ in Γ. It follows that $\langle a, b\rangle'$ is a normal subgroup of Γ, hence a normal subgroup of Γ_0. The index of $\langle a, b\rangle'$ in Γ_0 is

$$[\Gamma_0 : \langle a, b\rangle'] = \frac{[\Gamma : \langle a, b\rangle']}{[\Gamma : \Gamma_0]} = \frac{[\Gamma : \langle a, b\rangle] \cdot |\langle a, b\rangle^{ab}|}{[\Gamma : \Gamma_0]} = \frac{2 \cdot 64}{4} = 32.$$

Now, we apply [4, Corollary 2, Corollary 1] to get the stated amalgam decompositions of $\langle a, b\rangle' \cong \langle x, y\rangle'$, observing that

$$32 = [F_3 : F_{65}] = [F_{13} : F_{385}] = [F_5 : F_{129}] = [F_{17} : F_{513}].$$

The second claim follows similarly, using that $\langle x, y\rangle''$ has index $4096 = 8 \cdot 8 \cdot 64$ in $\langle x, y\rangle'$ by part (4) of this proposition.

(6) A relation in y and $xy^{-1}x$ induces a relation in $\psi(y) = b_2$ and $\psi(xy^{-1}x) = a_1 b_2^{-1} a_1 = b_3$, but $F_2 \cong \langle b_1, b_2\rangle < \langle b_1, b_2, b_3\rangle \cong F_3 < \Gamma$.

(7a) We compute

$$r = [x, yx^{-1}y] = -\frac{7}{25} + \frac{8}{75}i + \frac{32}{75}j + \frac{64}{75}k \in \langle x, y\rangle'$$

and

$$q = x^{-1}rx = [y, x^{-1}yx^{-1}] = -\frac{7}{25} - \frac{8}{25}i + \frac{16}{25}j + \frac{16}{25}k \in \langle x, y\rangle',$$

clearly both of norm 1. Since

$$r = -\frac{7}{25} + \frac{24}{25}\left(\frac{1}{9}i + \frac{4}{9}j + \frac{8}{9}k\right)$$

and

$$q = -\frac{7}{25} - \frac{24}{25}\left(\frac{1}{3}i - \frac{2}{3}j - \frac{2}{3}k\right),$$

they are both "rational" in the sense of [3, Definition 4.1], and by [3, Corollary 4.2] we have

$$F_2 \cong \langle r^2qr^4, r^4qr^2\rangle < \langle x, y\rangle' < \langle x, y\rangle.$$

(7b) We have $\psi([xy^{-1}x, y]) = b_3b_2b_3^{-1}b_2^{-1}$ and $\psi([y^{-1}, xy^{-1}x]) = b_2^{-1}b_3b_2b_3^{-1}$. The group $\langle b_3b_2b_3^{-1}b_2^{-1}, b_2^{-1}b_3b_2b_3^{-1}\rangle$ is free as a subgroup of $\langle b_2, b_3\rangle \cong F_2$, but not isomorphic to $\mathbb{Z}$, hence

$$\langle b_3b_2b_3^{-1}b_2^{-1}, b_2^{-1}b_3b_2b_3^{-1}\rangle \cong F_2.$$

By Lemma 3(9)

$$\langle [xy^{-1}x, y], [y^{-1}, xy^{-1}x]\rangle \cong F_2.$$

This is clearly a subgroup of $\langle x, y\rangle'$.

(7c) Let

$$q_1 := \frac{y^8}{5^4} \text{ and } q_2 := \frac{5^4(xy^{-1}x)^8}{3^8}.$$

First recall that $3^4, 5^4 \in \langle x, y\rangle$. It follows that $q_1, q_2 \in \langle x, y\rangle$. We have

$$\psi(q_1) = \psi(y^8) = b_2^8$$

and

$$\psi(q_2) = \psi((xy^{-1}x)^8) = b_3^8.$$

Since $\langle b_2^8, b_3^8\rangle \cong F_2$, Lemma 3(9) implies $\langle q_1, q_2\rangle \cong F_2$. It is easy to see that q_1 and q_2 both have norm 1. In particular $\langle q_1, q_2\rangle$ is a free subgroup of $\langle x, y\rangle \cap \mathbb{H}(\mathbb{Q})_1$, i.e. a free subgroup of $\langle x, y\rangle'$ using Proposition 10(2).

(8) This example is an illustration of the proof of Theorem 12(6). In [10, Section 4.1] we have shown that

$$\Gamma > \langle a_1a_2a_1a_2^{-1}, b_2^{-1}b_1^{-1}b_3^{-1}b_1\rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Using GAP ([6]) we check that

$$[\Gamma : \langle a, b, a_1a_2a_1a_2^{-1}\rangle] = [\Gamma : \langle a, b, b_2^{-1}b_1^{-1}b_3^{-1}b_1\rangle] = [\Gamma : \langle a, b\rangle] = 2,$$

hence $a_1a_2a_1a_2^{-1} \in \langle a, b\rangle$ and $b_2^{-1}b_1^{-1}b_3^{-1}b_1 \in \langle a, b\rangle$. Indeed, it is easy to check that

$$\psi(w_1) = ab^{-1}a^{-1}b^{-1}a^{-2}b^{-2} = a_1a_2a_1a_2^{-1}$$

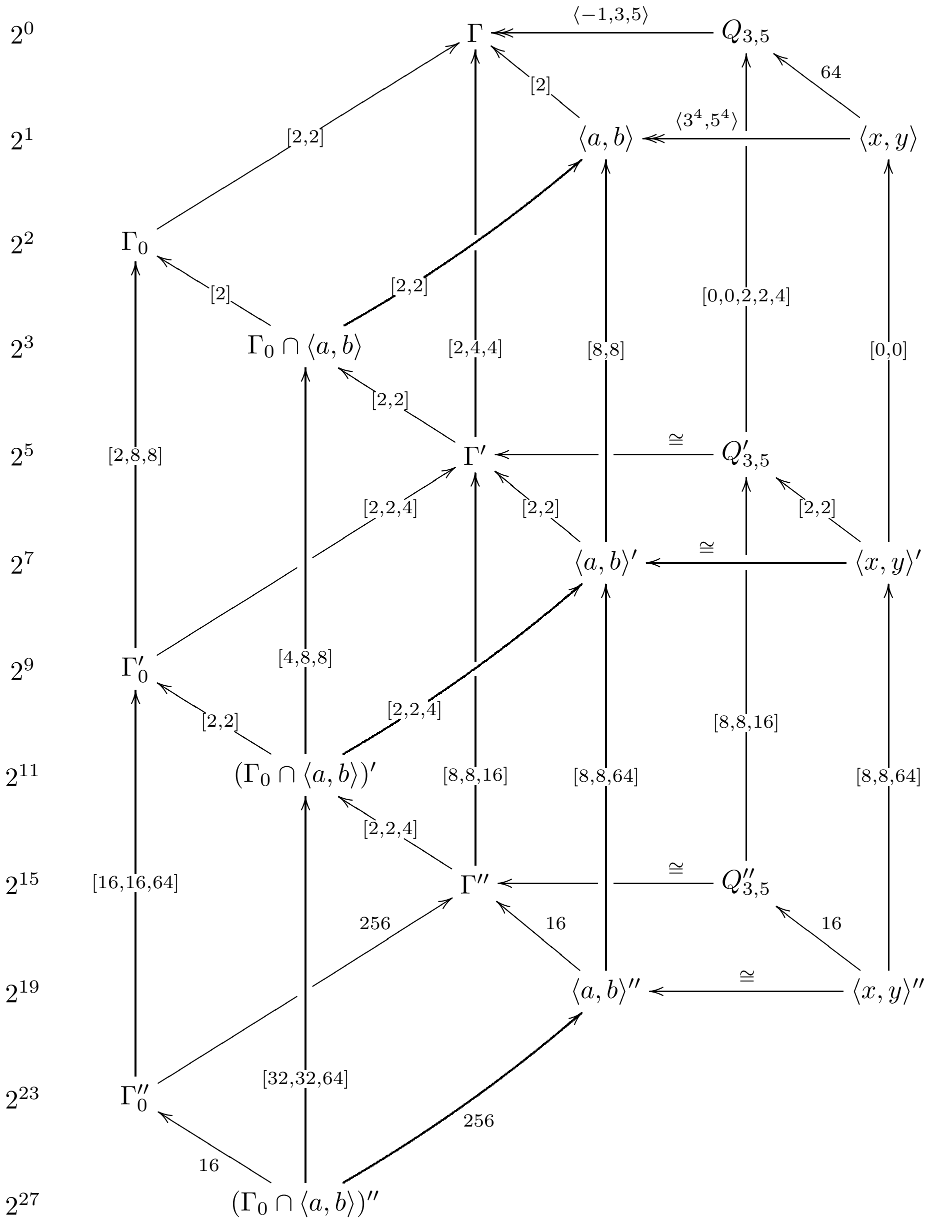
and

$$\psi(w_2) = b^{-1}a^2b^{-1}ab^{-1}ab^{-1}a^2 = b_2^{-1}b_1^{-1}b_3^{-1}b_1.$$

□

Remark 15. Using the presentation of $G = Q_{3,5}$ computed in the proof of Proposition 14(1), it is easy to check that G^{ab} and $(G/\langle\langle -1\rangle\rangle_G)^{ab}$ are not isomorphic. Indeed, $G^{ab} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, whereas $(G/\langle\langle -1\rangle\rangle_G)^{ab} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$. This shows that $-1 \notin G' = {Q_{3,5}}'$. In the same way, we have also checked that $-1 \notin {Q_{p,l}}'$, if $(p, l) = (3, 7)$, $(3, 11)$, $(5, 7)$, or $(5, 11)$, in particular Conjecture 6 and Conjecture 7 are true for these pairs (p, l).

See the big diagram below for the relations of some subgroups of Γ and $Q_{3,5}$ used in some previous proofs. In this diagram, all simple arrows are injective homomorphisms. A label of the form $[n_1, \dots, n_k]$ stands for the quotient group $\mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_k\mathbb{Z}$. The other labels of the injective homomorphisms are indices. The labels of the two surjective homomorphisms on top right indicate the corresponding kernels. The powers of 2 on the left are the indices of the subgroups of Γ on the same line.



4. The commutative case

In this section, we study the following simple question:

Question 16. *Given two distinct odd prime numbers p and l, are there commuting quaternions $x \in X_p$, $y \in X_l$?*

In the following, we will give general (negative and positive) answers to Question 16, except in the three cases $p \equiv 1, l \equiv 7 \pmod 8$, $p \equiv 7, l \equiv 1 \pmod 8$ and $p, l \equiv 7 \pmod 8$, where the situation seems to be quite complicated.

Let $T_{p,l}$ be the set $Q_{p,l} \cap \mathbb{H}(\mathbb{Z})$, i.e.

$$\begin{aligned} T_{p,l} := \{x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})\,; \quad & |x|^2 = p^r l^s,\ r, s \in \mathbb{N}_0; \\ & x_0 \text{ odd}, x_1, x_2, x_3 \text{ even, if } |x|^2 \equiv 1 \pmod 4\,; \\ & x_1 \text{ even}, x_0, x_2, x_3 \text{ odd, if } |x|^2 \equiv 3 \pmod 4\}\,. \end{aligned}$$

Then clearly $X_p \subset T_{p,l}$ and $X_l \subset T_{p,l}$. Note that $x, y \in T_{p,l}$ commute, if and only if $\psi_{p,l}(x), \psi_{p,l}(y)$ commute in $\Gamma_{p,l}$, but we will directly work with quaternions here and not use the group $\Gamma_{p,l}$ anymore.

Let $x = x_0 + x_1 i + x_2 j + x_3 k \in T_{p,l}$ such that $(x_1, x_2, x_3) \neq (0,0,0)$. We can write it as

$$x = x_0 + z_x(c_1 i + c_2 j + c_3 k)$$

such that $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime and $z_x \in \mathbb{Z} \setminus \{0\}$. Up to multiplication by -1, the integers c_1, c_2, c_3, z_x are uniquely determined by x. Therefore we can define the number

$$n(x) := c_1^2 + c_2^2 + c_3^2 \in \mathbb{N}.$$

Moreover, if $x = x_0 \in T_{p,l} \cap \mathbb{Q}^* = T_{p,l} \cap \mathbb{Z}$, we define $n(x) := 0$.

Remark 17. It is easy to show that $n(x) \equiv 1, 2, 3, 5, 6 \pmod 8$, if $n(x) \neq 0$.

The function $n : T_{p,l} \to \mathbb{N}_0$ is a useful invariant for commutativity:

Lemma 18. *Let p, l be two distinct odd prime numbers and let $x, y \in T_{p,l}$ be commuting quaternions such that $x, y \notin \mathbb{Q}^*$. Then $n(x) = n(y) \neq 0$.*

Proof. This follows directly from the basic fact (see [8, Section 3] or [11, Lemma 12]) that two quaternions $x = x_0 + x_1 i + x_2 j + x_3 k$ and $y = y_0 + y_1 i + y_2 j + y_3 k$ in $\mathbb{H}(\mathbb{Q}) \setminus \mathbb{Q}$ commute, if and only if $\mathbb{Q}(x_1, x_2, x_3) = \mathbb{Q}(y_1, y_2, y_3)$. ☐

Lemma 18 will be used in Proposition 22 to show that certain pairs $x \in X_p$, $y \in X_l$ cannot commute. To apply this lemma we have to get some knowledge on n. This is done in Lemma 20 for the cases where $|x|^2 \not\equiv 1 \pmod 8$. First we state an auxiliary very basic lemma:

Lemma 19. *If $x_0 \in \mathbb{Z}$ is odd, then $x_0^2 \equiv 1 \pmod 8$.*

Proof. Let $x_0 = 1 + 2t$ for some $t \in \mathbb{Z}$. Then $x_0^2 = 1 + 4t(1+t) \equiv 1 \pmod 8$, since $t(1+t)$ is always even. ☐

Lemma 20. *Let p be an odd prime number and let $x \in X_p$.*

1. *If $p \equiv 5 \pmod 8$, then $n(x)$ is odd.*
2. *If $p \equiv 3 \pmod 8$, then $n(x) \equiv 2 \pmod 8$.*
3. *If $p \equiv 7 \pmod 8$, then $n(x) \equiv 6 \pmod 8$.*

Proof. First note that $X_p \cap \mathbb{Q} = \emptyset$. Write $x = x_0 + z_x(c_1 i + c_2 j + c_3 k)$ as in the definition of n.

(1) Since $|x|^2 \equiv 1 \pmod 4$, x_0 is odd and z_x is even. We have
$$p = |x|^2 = x_0^2 + z_x^2 n(x) \equiv 5 \pmod 8.$$
If $n(x)$ would be even, then $z_x^2 n(x) \equiv 0 \pmod 8$, hence $x_0^2 \equiv 5 \pmod 8$, contradicting Lemma 19.

(2) Here, $|x|^2 \equiv 3 \pmod 4$, hence x_0, z_x are odd and $n(x)$ is even (as a sum of two odd squares and one even square). We have
$$p = |x|^2 = x_0^2 + z_x^2 n(x) \equiv 3 \pmod 8,$$
hence $z_x^2 n(x) \equiv 2 \pmod 8$ using Lemma 19. Since $z_x^2 \equiv 1 \pmod 8$ by Lemma 19, it follows that $n(x) \equiv 2 \pmod 8$.

(3) The proof is completely analogous to the proof of part (2).

□

Remark 21. We will later see from Table 4 that all of the possibilities $n(x) \equiv 1, 3, 5 \pmod 8$ (in view of Remark 17 and Lemma 20(1)) can be realized if $p \equiv 5 \pmod 8$. Moreover, in the case $p \equiv 1 \pmod 8$ not treated in Lemma 20, all possibilities $n(x) \equiv 1, 2, 3, 5, 6 \pmod 8$ can be realized.

Proposition 22. *Let p, l be two distinct odd prime numbers. Suppose that $p, l \not\equiv 1 \pmod 8$ and $p \not\equiv l \pmod 8$. Then there are no commuting quaternions $x \in X_p$, $y \in X_l$.*

Proof. This follows directly from Lemma 18, using Lemma 20. □

To obtain positive answers to Question 16, we will use some known results on prime numbers of the form $r^2 + ms^2$, first for $m = 1$ and $m = 2$ in Proposition 24, later for $m = 6$ and $m = 22$ in Proposition 25.

Lemma 23. *(Fermat, see* [2, (1.1)]*) Let p be an odd prime number. There are $x_0, x_1 \in \mathbb{Z}$ such that $x_0^2 + x_1^2 = p$, if and only if $p \equiv 1 \pmod 4$. There are $x_0, z \in \mathbb{Z}$ such that $x_0^2 + 2z^2 = p$, if and only if $p \equiv 1, 3 \pmod 8$.*

This lemma can be applied as follows:

Proposition 24. *Let p, l be two distinct odd prime numbers. Suppose that either $p, l \equiv 1 \pmod 4$ or that $p, l \equiv 1, 3 \pmod 8$. Then there are commuting quaternions $x \in X_p$, $y \in X_l$.*

Proof. If $p, l \equiv 1 \pmod 4$ then by Lemma 23, there are x_0, y_0 odd, x_1, y_1 even, such that
$$x_0^2 + x_1^2 = p \quad \text{and} \quad y_0^2 + y_1^2 = l.$$
Now we take the commuting quaternions $x = x_0 + x_1 i \in X_p$ and $y = y_0 + y_1 i \in X_l$.

If $p \equiv 1 \pmod 8$, then by Lemma 23 there are $x_0, z \in \mathbb{Z}$ such that $x_0^2 + 2z^2 = p$. It follows that x_0 is odd, hence $2z^2 \equiv 0 \pmod 8$ by Lemma 19 and z is even (but non-zero). We choose
$$x := x_0 + z(j + k) \in X_p,$$
in particular $|x|^2 = x_0^2 + 2z^2 = p$ and $n(x) = 2$.

If $p \equiv 3 \pmod 8$, then again $x_0^2 + 2z^2 = p$ by Lemma 23, but here x_0 and z are both odd, and we take as above

$$x := x_0 + z(j + k) \in X_p,$$

such that $|x|^2 = p$ and $n(x) = 2$.

In the same way we construct

$$y := y_0 + z_y(j + k) \in X_l$$

such that $|y|^2 = l \equiv 1, 3 \pmod 8$ and $n(y) = 2$. Clearly $xy = yx$ by construction. □

We illustrate the results of Proposition 22 and Proposition 24 in Table 1 for p, l taken modulo 8 ("+" means that there are always commuting quaternions $x \in X_p$, $y \in X_l$, "$-$" means that there are never such commuting quaternions, "$\pm$" means that both cases happen).

(mod 8)	$l \equiv 1$	3	5	7
$p \equiv 1$	$+$	$+$	$+$	$\pm$
3	$+$	$+$	$-$	$-$
5	$+$	$-$	$+$	$-$
7	$\pm$	$-$	$-$	$\pm$

Table 1. Existence and non-existence of commuting quaternions

In the three cases $p \equiv 1, l \equiv 7 \pmod 8$, $p \equiv 7, l \equiv 1 \pmod 8$ and $p, l \equiv 7 \pmod 8$ excluded from Proposition 22 and Proposition 24, it seems to be more difficult to decide in general by congruence conditions whether or not there are commuting quaternions $x \in X_p$, $y \in X_l$. We illustrate this with results for some subcases of $p, l \equiv 7 \pmod 8$ and $p \equiv 1, l \equiv 7 \pmod 8$:

Proposition 25. *Let p, l be two distinct odd prime numbers.*

(1) *Suppose that $p, l \equiv 7 \pmod{24}$ or $p, l \equiv 15, 23, 31, 47, 71 \pmod{88}$. Then there are commuting quaternions $x \in X_p$, $y \in X_l$.*

(2) *Suppose that $p \equiv 1, l \equiv 7 \pmod{24}$ or $p \equiv 1, 9, 25, 49, 81 \pmod{88}$, $l \equiv 15, 23, 31, 47, 71 \pmod{88}$. Then there are commuting quaternions $x \in X_p$, $y \in X_l$.*

Proof. (1) A prime number p is of the form $x_0^2 + 6z^2$, if and only if $p \equiv 1, 7 \pmod{24}$ (see [2] for a proof, but beware of the misprint in [2, (2.28)] stating the condition (mod 12) instead of (mod 24)). Let $p, l \equiv 7 \pmod{24}$. There are $x_0, z \in \mathbb{Z}$ such that

$$p = x_0^2 + 6z^2 = x_0^2 + (2z)^2 + z^2 + z^2.$$

It follows that x_0 and z are odd. Take

$$x := x_0 + z(2i + j + k) \in X_p,$$

hence $|x|^2 = x_0^2 + 6z^2 = p$ and $n(x) = 6$. In the same way, we choose

$$y := y_0 + z_y(2i + j + k) \in X_l$$

such that $|y|^2 = y_0^2 + 6z_y^2 = l$, $n(y) = 6$ and $xy = yx$.

If $p, l \equiv 15, 23, 31, 47, 71 \pmod{88}$, then we can give a similar proof, using the fact (see [2, (2.28)]) that a prime number p is of the form $x_0^2+22z^2$, if and only if $p \equiv 1, 9, 15, 23, 25, 31, 47, 49, 71, 81 \pmod{88}$. Observe that $22z^2 \equiv 6 \pmod 8$ if z is odd, and $22z^2 \equiv 0 \pmod 8$ if z is even. Therefore z is odd here. We take

$$x := x_0 + z(2i + 3j + 3k) \in X_p,$$

hence $|x|^2 = x_0^2 + 22z^2 = p$ and $n(x) = 22$. Now, we are done by the analogous construction for y.

(2) Let $p \equiv 1 \pmod{24}$. The proof is similar as in (1), but here $p = x_0^2 + 6z^2$ for some x_0 odd, $z =: 2\tilde{z}$ even (and non-zero), hence

$$p = x_0^2 + 24\tilde{z}^2 = x_0^2 + (4\tilde{z})^2 + (2\tilde{z})^2 + (2\tilde{z})^2$$

and we choose

$$x := x_0 + \tilde{z}(4i + 2j + 2k)$$

which commutes with $2i + j + k$.

Let $p \equiv 1, 9, 25, 49, 81 \pmod{88}$. Similarly as above we can choose

$$x := x_0 + \tilde{z}(4i + 6j + 6k)$$

of norm $x_0^2 + 22(2\tilde{z})^2$ commuting with $2i + 3j + 3k$.

□

The two quaternions $x = 1 + 2i$ and $y = 1 + 4k$ do not commute, but $n(x) = n(y) = 1$, so the converse of Lemma 18 is certainly not true. However, the function n determines the *existence* of commuting quaternions of given norms.

Proposition 26. *Let p, l be two distinct odd prime numbers and let $x, y \in T_{p,l}$ such that $n(x) = n(y)$. Then there are commuting quaternions $\hat{x}, \hat{y} \in T_{p,l}$ such that $|\hat{x}|^2 = |x|^2$ and $|\hat{y}|^2 = |y|^2$.*

Proof. If $n(x) = n(y) = 0$, then $x = x_0$, $y = y_0$ and we can take $\hat{x} := x$, $\hat{y} := y$.

Now suppose that $n(x) = n(y) \neq 0$. Write

$$x = x_0 + x_1 i + x_2 j + x_3 k = x_0 + z_x(c_1 i + c_2 j + c_3 k)$$

such that $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime and $z_x \in \mathbb{Z} \setminus \{0\}$, and write

$$y = y_0 + y_1 i + y_2 j + y_3 k = y_0 + z_y(d_1 i + d_2 j + d_3 k)$$

such that $d_1, d_2, d_3 \in \mathbb{Z}$ are relatively prime and $z_y \in \mathbb{Z} \setminus \{0\}$. Then by assumption

$$c_1^2 + c_2^2 + c_3^2 = n(x) = n(y) = d_1^2 + d_2^2 + d_3^2.$$

If $|x|^2 \equiv 1 \pmod 4$, then z_x is even (and non-zero). Let

$$\hat{x} := x_0 + z_x(d_1 i + d_2 j + d_3 k) \in T_{p,l}.$$

Then

$$|\hat{x}|^2 = x_0^2 + z_x^2 n(y) = x_0^2 + z_x^2 n(x) = |x|^2$$

and $\hat{x}$ commutes with $\hat{y} := y$.

If $|y|^2 \equiv 1 \pmod 4$, then we imitate this proof interchanging x and y.

If $|x|^2, |y|^2 \equiv 3 \pmod 4$, then x_0, z_x, c_2, c_3, y_0, z_y, d_2, d_3 are odd, c_1, d_1 are even and we can take

$$\hat{x} := x_0 + z_x(d_1 i + d_2 j + d_3 k) \in T_{p,l}$$

and $\hat{y} := y$ as above. □

For an odd prime number p we consider the set $n(X_p) = \{n(x) : x \in X_p\}$. This is a finite subset of $\mathbb{N}$ satisfying $\max n(X_p) \leq p - 1$.

Corollary 27. *Let p, l be two distinct odd prime numbers. There are commuting quaternions $x \in X_p$, $y \in X_l$, if and only if $n(X_p) \cap n(X_l) \neq \emptyset$.*

Proof. We combine Lemma 18 and Proposition 26 using that $(X_p \cup X_l) \subset T_{p,l}$ and $(X_p \cup X_l) \cap \mathbb{Q} = \emptyset$. □

For $p < 200$, we list all sets $n(X_p)$ in Table 3 in the Appendix. It shows for example that there are prime numbers $p, l \equiv 7 \pmod 8$ excluded from Proposition 25 with commuting quaternions $x \in X_p$, $y \in X_l$. For example take $p = 47$ and $l = 167$ (such that $l \equiv 23 \pmod{24}$ and $l \equiv 79 \pmod{88}$), where $n(X_{47}) \cap n(X_{167}) = \{46\}$, and take commuting quaternions $x = 1 + 6i + j + 3k$ and $y = 11 + 6i + j + 3k$ such that $|x|^2 = 47$, $|y|^2 = 167$ and $n(x) = n(y) = 46$.

In Table 5 we also list $n(X_p)$ for all prime numbers $p < 1000$ satisfying $p \equiv 23 \pmod{24}$ and $p \equiv 7, 39, 63, 79, 87 \pmod{88}$, that is for all prime numbers $p \equiv 7 \pmod 8$, $p < 1000$, excluded from Proposition 25(1).

Lemma 28. *Let p be an odd prime number and $m \in n(X_p)$. Then there exist $r, s \in \mathbb{N}$ such that $p = r^2 + ms^2$.*

Proof. Let $x = x_0 + x_1 i + x_2 j + x_3 k \in X_p$ such that $n(x) = m$. Since $n(x) = (x_1^2 + x_2^2 + x_3^2)/t^2$ for some $t \in \mathbb{N}$, we have $p = x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 + mt^2$, i.e. we can take $s = t$ and $r = |x_0|$. □

The converse of Lemma 28 is true in some special cases as seen in the proofs of Proposition 24 and Proposition 25. However it is not true in general, for example take $p = 7$, $m = 3$, $r = 2$, $s = 1$ and observe that $m \notin n(X_7) = \{6\}$.

Corollary 29. *Let p, l be two distinct odd prime numbers. If there are commuting quaternions $x \in X_p$, $y \in X_l$, then there exists an $m \in \mathbb{N}$ and $r_1, r_2, s_1, s_2 \in \mathbb{N}$ such that $p = r_1^2 + ms_1^2$ and $l = r_2^2 + ms_2^2$.*

Proof. Combine Corollary 27 and Lemma 28. □

For p an odd prime number, let $n(X_p)_{min}$ be the smallest element in $n(X_p)$. From what we have seen, it immediately follows that $n(X_p)_{min} = 1$, if $p \equiv 1 \pmod 4$, $n(X_p)_{min} = 2$, if $p \equiv 3 \pmod 8$, and $n(X_p)_{min} = 6$, if $p \equiv 7 \pmod{24}$. In the remaining case $p \equiv 23 \pmod{24}$, there seems to be no upper bound for $n(X_p)_{min}$. We compute for example $n(X_{23})_{min} = 14$, $n(X_{47})_{min} = 22$, $n(X_{167})_{min} = 46$, $n(X_{503})_{min} = 62$, $n(X_{1223})_{min} = 134$, $n(X_{1823})_{min} = 142$, $n(X_{1847})_{min} = 166$, $n(X_{4703})_{min} = 214$, $n(X_{8543})_{min} = 262$, $n(X_{9743})_{min} = 334$. This phenomenon makes it difficult to answer Question 16 in full generality.

5. Appendix: some lists

In the following list (Table 2), we give some examples of pairs $x \in X_p$, $y \in X_l$ such that the index of $\langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle$ in $\Gamma_{p,l}$ is finite (in particular Theorem 12 can be applied). This index is given in the fifth column of the list. We have used GAP ([6]) for the computations. We see that for $(p,l) = (3,5)$ there are only two possibilities for the index and abelianization. For the other pairs (p,l), we have therefore only included some "typical" examples to keep the list reasonably short. It also happened that for some non-commuting pairs x, y (for example $x = 1+j+k$, $y = 1+6i+2k$, or if p, l are large) we were not able to compute the index and the abelianization. In these cases we do not know if the index is indeed infinite or finite (but perhaps very large or difficult to compute). Recall that for *free* groups $\langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle$, the index would be infinite.

p	l	x	y	index	$\langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle^{ab}$
3	5	$1+j+k$	$1+2i$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$1+2j$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+2k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j-k$	$1+2i$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j-k$	$1+2j$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j-k$	$1+2k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	7	$1+j+k$	$1+2i+j+k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$1+2i+j-k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	11	$1+j+k$	$1+j+3k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+j-3k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
3	13	$1+j+k$	$1+2i+2j+2k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$3+2i$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$3+2j$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	17	$1+j+k$	$1+4i$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$1+4j$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$3+2i+2j$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+2j-2k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
3	19	$1+j+k$	$1+4i+j+k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$1+4i+j-k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+3j-3k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+j-3k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
3	23	$1+j+k$	$1+2i+3j+3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$1+2i+3j-3k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+j+k$	$3+2i+j+3k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	29	$1+j+k$	$3+4i+2j$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+2i+4j$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$5+2i$	160	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$5+2j$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	31	$1+j+k$	$1+2i+j+5k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$1+2i+j-5k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+j+k$	$5+2i+j-k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+2i+3j+3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$3+2i+3j-3k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$

3	37	$1+j+k$	$1+6i$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$1+6j$	22	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+2i+4j+4k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$1+4i+2j+4k$	22	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+4i+2j-4k$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$5+2i+2j+2k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$5+2i+2j-2k$	160	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
3	41	$1+j+k$	$1+2j+6k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$1+2j-6k$	32	$\mathbb{Z}_8 \times \mathbb{Z}_{128}$
		$1+j+k$	$1+2i+6k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+j+k$	$3+4i+4j$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$3+4i-4k$	32	$\mathbb{Z}_8 \times \mathbb{Z}_{128}$
		$1+j+k$	$5+4j$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
3	43	$1+j+k$	$3+3j-5k$	32	$\mathbb{Z}_8 \times \mathbb{Z}_{128}$
		$1+j+k$	$5+3j-3k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+4i+j+5k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$1+4i+j-5k$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$5+4i+j+k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$5+4i+j-k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+4i+3j+3k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$3+4i+3j-3k$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
3	47	$1+j+k$	$3+2i+3j-5k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$5+2i+3j-3k$	1920	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+j+k$	$1+6i+j+3k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_{72}$
		$1+j+k$	$1+6i+j-3k$	96	$\mathbb{Z}_8 \times \mathbb{Z}_{80}$
		$1+j+k$	$3+6i+j+k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
3	53	$1+j+k$	$1+4j+6k$	22	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+2i+2j+6k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$7+2k$	22	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	59	$1+j+k$	$5+4i+3j+3k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
3	61	$1+j+k$	$3+4j+6k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$3+4j-6k$	80	$\mathbb{Z}_{24} \times \mathbb{Z}_{40}$
		$1+j+k$	$5+6k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$5+4i+2j+4k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
3	67	$1+j+k$	$1+4i+j+7k$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$1+4i+j-7k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$5+4i+j-5k$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
3	71	$1+j+k$	$3+6i+j+5k$	96	$\mathbb{Z}_8 \times \mathbb{Z}_{80}$
3	73	$1+j+k$	$1+6j-6k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+k$	$1+6i+6k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+2i+2j+8k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+j+k$	$1+2i+2j-8k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$1+8i+2j+2k$	64	$\mathbb{Z}_8 \times \mathbb{Z}_{256}$
		$1+j+k$	$3+8i$	64	$\mathbb{Z}_8 \times \mathbb{Z}_{256}$
		$1+j+k$	$3+8j$	32	$\mathbb{Z}_8 \times \mathbb{Z}_{128}$
		$1+j+k$	$5+4i+4j+4k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$

		$1+j+k$	$5+4i+4j-4k$	640	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$7+2i+2j+4k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+k$	$7+2i+2j-4k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+j+k$	$7+4i+2j+2k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+j+k$	$7+4i+2j-2k$	672	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
3	79	$1+j+k$	$3+6i+3j+5k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_{72}$
		$1+j+k$	$5+2i+j-7k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+k$	$5+6i+3j+3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
3	83	$1+j+k$	$1+8i+3j+3k$	64	$\mathbb{Z}_8 \times \mathbb{Z}_{256}$
3	89	$1+j+k$	$3+4i+8k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
5	7	$1+2i$	$1+2i+j+k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2j$	$1+2i+j+k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
5	11	$1+2i$	$1+j+3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2i$	$3+j+k$	48	$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$
		$1+2j$	$1+j+3k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
		$1+2j$	$1+3j+k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
5	13	$1+2i$	$1+2i+2j+2k$	16	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
		$1+2i$	$3+2j$	96	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
5	17	$1+2i$	$1+4j$	32	$\mathbb{Z}_{16} \times \mathbb{Z}_{64}$
		$1+2i$	$3+2i+2j$	8	$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$
		$1+2i$	$3+2j+2k$	192	$\mathbb{Z}_{16} \times \mathbb{Z}_{64}$
5	19	$1+2i$	$1+4i+j+k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2i$	$1+3j+3k$	96	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2i$	$3+j+3k$	48	$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$
		$1+2j$	$1+4i+j+k$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+2j$	$1+3j+3k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
		$1+2j$	$3+3j+k$	24	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
5	23	$1+2i$	$1+2i+3j+3k$	96	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2i$	$3+2i+j+3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2j$	$1+2i+3j+3k$	112	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
		$1+2j$	$3+2i+3j+k$	24	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
5	29	$1+2i$	$3+4i+2j$	8	$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$
		$1+2i$	$3+2i+4j$	32	$\mathbb{Z}_{16} \times \mathbb{Z}_{64}$
		$1+2i$	$3+2j+4k$	96	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
		$1+2i$	$5+2j$	8	$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$
5	31	$1+2i$	$1+2i+j+5k$	224	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2j$	$1+2i+j+5k$	240	$\mathbb{Z}_8 \times \mathbb{Z}_{56}$
		$1+2j$	$3+2i+3j+3k$	1344	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
5	37	$1+2i$	$1+4i+2j+4k$	8	$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$
5	41	$1+2i$	$1+2j+6k$	16	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
		$1+2i$	$1+2i+6k$	192	$\mathbb{Z}_{16} \times \mathbb{Z}_{48}$
		$1+2i$	$3+4j+4k$	768	$\mathbb{Z}_{16} \times \mathbb{Z}_{256}$
		$1+2i$	$3+4i+4k$	32	$\mathbb{Z}_{16} \times \mathbb{Z}_{64}$
		$1+2i$	$5+4j$	32	$\mathbb{Z}_{16} \times \mathbb{Z}_{64}$
5	43	$1+2j$	$3+3j+5k$	24	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2j$	$3+5j+3k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$

		$1+2j$	$3+4i+3j+3k$	24	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
5	47	$1+2i$	$3+2i+5j-3k$	288	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
		$1+2j$	$3+2i+5j+3k$	112	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
7	11	$1+2i+j+k$	$1+j+3k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+2i+j+k$	$1+j-3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2i+j+k$	$3+j+k$	20	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+2i+j+k$	$3+j-k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
7	13	$1+2i+j+k$	$1+2i+2j+2k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2i+j+k$	$1-2i+2j+2k$	32	$\mathbb{Z}_{24} \times \mathbb{Z}_{48}$
		$1+2i+j+k$	$3+2i$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2i+j+k$	$3+2j$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
7	17	$1+2i+j+k$	$1+4i$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+2i+j+k$	$1+4j$	192	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+2i+j+k$	$3+2i+2j$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
		$1+2i+j+k$	$3+2i-2j$	2736	$\mathbb{Z}_8 \times \mathbb{Z}_{40}$
		$1+2i+j+k$	$3+2j+2k$	16	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$1+2i+j+k$	$3+2j-2k$	192	$\mathbb{Z}_8 \times \mathbb{Z}_{96}$
7	19	$1+2i+j+k$	$1+4i+j+k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2i+j+k$	$1+4i+j-k$	48	$\mathbb{Z}_{24} \times \mathbb{Z}_{40}$
		$1+2i+j+k$	$1+4i-j-k$	160	$\mathbb{Z}_{24} \times \mathbb{Z}_{48}$
		$1+2i+j+k$	$1+3j+3k$	160	$\mathbb{Z}_{24} \times \mathbb{Z}_{48}$
		$1+2i+j+k$	$3+j+3k$	48	$\mathbb{Z}_{24} \times \mathbb{Z}_{40}$
		$1+2i+j+k$	$3+j-3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
7	23	$1+2i+j+k$	$3+2i+3j-k$	192	$\mathbb{Z}_8 \times \mathbb{Z}_{96}$
7	29	$1+2i+j+k$	$3+4i+2j$	48	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+2i+j+k$	$3+2j-4k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+2i+j+k$	$5+2j$	48	$\mathbb{Z}_8 \times \mathbb{Z}_8$
7	31	$1+2i+j+k$	$1+2i-j+5k$	240	$\mathbb{Z}_{24} \times \mathbb{Z}_{56}$
		$1+2i+j+k$	$3+2i+3j-3k$	240	$\mathbb{Z}_{24} \times \mathbb{Z}_{56}$
11	13	$1+j+3k$	$1+2i+2j-2k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+3k$	$3+2k$	72	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$3+j+k$	$1+2i+2j-2k$	288	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$3+j+k$	$3+2j$	72	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$3+j-k$	$1+2i-2j+2k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
11	17	$1+j+3k$	$1+4j$	192	$\mathbb{Z}_8 \times \mathbb{Z}_{96}$
		$1+j+3k$	$1+4k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$1+j+3k$	$3+2i+2k$	72	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+3k$	$3+2j+2k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$3+j+k$	$3+2i+2j$	144	$\mathbb{Z}_8 \times \mathbb{Z}_{24}$
11	19	$3+j+k$	$1+4i+j+k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_{192}$
		$3+j+k$	$3+j-3k$	96	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
11	23	$1+j+3k$	$2+i+3j+3k$	80	$\mathbb{Z}_8 \times \mathbb{Z}_{72}$
		$1+j+3k$	$2+i+3j-3k$	56	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$3+j+k$	$2+i+3j-3k$	336	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$3+j+k$	$2+3i+j-3k$	96	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$

11	29	$1+j+3k$	$3+2j+4k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
		$1+j+3k$	$5+2k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
11	31	$3+j+k$	$1+2i+j-5k$	2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
11	37	$1+j+3k$	$1+2i+4j-4k$	56	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$1+j+3k$	$1+4i+2j+4k$	336	$\mathbb{Z}_8 \times \mathbb{Z}_{80}$
11	41	$3+j+k$	$1+2j+6k$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$3+j+k$	$5+4j$	8	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
13	17	$3+2i$	$3+2i+2j$	288	$\mathbb{Z}_{16} \times \mathbb{Z}_{32}$
13	19	$1+2i+2j+2k$	$1+4i+j-k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
		$1+2i+2j+2k$	$3+j-3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
13	23	$3+2i$	$3+2i+j+3k$	1152	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$3+2j$	$1+2i+3j+3k$	24	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
13	29	$3+2i$	$3+2i+4j$	384	$\mathbb{Z}_{32} \times \mathbb{Z}_{64}$
13	31	$1+2i+2j+2k$	$3+2i+3j-3k$	4	$\mathbb{Z}_8 \times \mathbb{Z}_{48}$
17	19	$1+4i$	$1+4i+j+k$	1920	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
		$3+2i+2j$	$3+j+3k$	72	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
		$3+2j+2k$	$1+4i+j-k$	576	$\mathbb{Z}_8 \times \mathbb{Z}_{288}$
		$3+2j+2k$	$1+3j-3k$	576	$\mathbb{Z}_8 \times \mathbb{Z}_{288}$
		$3+2j+2k$	$3+j+3k$	96	$\mathbb{Z}_8 \times \mathbb{Z}_{64}$
19	23	$3+j+3k$	$1+2i+3j-3k$	48	$\mathbb{Z}_8 \times \mathbb{Z}_{32}$
		$3+j+3k$	$3+2i+j-3k$	24	$\mathbb{Z}_8 \times \mathbb{Z}_{16}$

Table 2: Some examples where $[\Gamma_{p,l} : \langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle] < \infty$

By Corollary 27, the following table can be used to answer Question 16 for all pairs $p, l < 200$.

p	$n(X_p)$	p	$n(X_p)$
3	$\{2\}$	101	$\{1,5,13,19,25\}$
5	$\{1\}$	103	$\{6,22,54,78,94,102\}$
7	$\{6\}$	107	$\{2,26,58,82,98,106\}$
11	$\{2,10\}$	109	$\{1,3,21,25,27\}$
13	$\{1,3\}$	113	$\{1,2,22,26\}$
17	$\{1,2\}$	127	$\{6,14,46,78,102,118,126\}$
19	$\{2,10,18\}$	131	$\{2,10,50,82,106,122,130\}$
23	$\{14,22\}$	137	$\{1,2,14,22,34\}$
29	$\{1,5\}$	139	$\{2,10,18,58,90,114,130,138\}$
31	$\{6,22,30\}$	149	$\{1,17,25,35,37\}$
37	$\{1,3,9\}$	151	$\{6,14,30,70,102,126,142,150\}$
41	$\{1,2,10\}$	157	$\{1,3,9,19,27,33,37\}$
43	$\{2,18,34,42\}$	163	$\{2,18,42,82,114,138,154,162\}$
47	$\{22,38,46\}$	167	$\{46,86,118,142,158,166\}$
53	$\{1,11,13\}$	173	$\{1,13,37,41,43\}$
59	$\{2,10,34,50,58\}$	179	$\{2,10,58,98,130,154,170,178\}$
61	$\{1,3,9,13\}$	181	$\{1,3,5,25,33,43,45\}$
67	$\{2,18,42,58,66\}$	191	$\{22,70,110,142,166,182,190\}$
71	$\{22,46,62,70\}$	193	$\{1,2,3,6,9,18,42,46\}$
73	$\{1,2,3,6,18\}$	197	$\{1,19,29,37,43,49\}$
79	$\{6,30,54,70,78\}$	199	$\{6,22,30,78,118,150,174,190,198\}$
83	$\{2,34,58,74,82\}$		
89	$\{1,2,5,10,22\}$		
97	$\{1,2,3,6,18,22\}$		

TABLE 3. $n(X_p)$ for prime numbers $2 < p < 200$

Table 4 contains the same information as Table 3, but is sorted by $p \pmod 8$. It is a good illustration of Lemma 20.

p	$n(X_p)$	p	$n(X_p)$
3	$\{2\}$	7	$\{6\}$
11	$\{2, 10\}$	23	$\{14, 22\}$
19	$\{2, 10, 18\}$	31	$\{6, 22, 30\}$
43	$\{2, 18, 34, 42\}$	47	$\{22, 38, 46\}$
59	$\{2, 10, 34, 50, 58\}$	71	$\{22, 46, 62, 70\}$
67	$\{2, 18, 42, 58, 66\}$	79	$\{6, 30, 54, 70, 78\}$
83	$\{2, 34, 58, 74, 82\}$	103	$\{6, 22, 54, 78, 94, 102\}$
107	$\{2, 26, 58, 82, 98, 106\}$	127	$\{6, 14, 46, 78, 102, 118, 126\}$
131	$\{2, 10, 50, 82, 106, 122, 130\}$	151	$\{6, 14, 30, 70, 102, 126, 142, 150\}$
139	$\{2, 10, 18, 58, 90, 114, 130, 138\}$	167	$\{46, 86, 118, 142, 158, 166\}$
163	$\{2, 18, 42, 82, 114, 138, 154, 162\}$	191	$\{22, 70, 110, 142, 166, 182, 190\}$
179	$\{2, 10, 58, 98, 130, 154, 170, 178\}$	199	$\{6, 22, 30, 78, 118, 150, 174, 190, 198\}$
5	$\{1\}$	17	$\{1, 2\}$
13	$\{1, 3\}$	41	$\{1, 2, 10\}$
29	$\{1, 5\}$	73	$\{1, 2, 3, 6, 18\}$
37	$\{1, 3, 9\}$	89	$\{1, 2, 5, 10, 22\}$
53	$\{1, 11, 13\}$	97	$\{1, 2, 3, 6, 18, 22\}$
61	$\{1, 3, 9, 13\}$	113	$\{1, 2, 22, 26\}$
101	$\{1, 5, 13, 19, 25\}$	137	$\{1, 2, 14, 22, 34\}$
109	$\{1, 3, 21, 25, 27\}$	193	$\{1, 2, 3, 6, 9, 18, 42, 46\}$
149	$\{1, 17, 25, 35, 37\}$		
157	$\{1, 3, 9, 19, 27, 33, 37\}$		
173	$\{1, 13, 37, 41, 43\}$		
181	$\{1, 3, 5, 25, 33, 43, 45\}$		
197	$\{1, 19, 29, 37, 43, 49\}$		

TABLE 4. $n(X_p)$ for prime numbers $2 < p < 200$, sorted by $p \pmod 8$

In Table 5 we list $n(X_p)$ for all prime numbers $p < 1000$ satisfying $p \equiv 23 \pmod{24}$ and $p \equiv 7, 39, 63, 79, 87 \pmod{88}$.

p	$n(X_p)$
167	$\{46, 86, 118, 142, 158, 166\}$
239	$\{14, 70, 118, 158, 190, 214, 230, 238\}$
263	$\{38, 94, 142, 182, 214, 238, 254, 262\}$
359	$\{14, 70, 134, 190, 238, 278, 310, 334, 350, 358\}$
431	$\{14, 70, 142, 206, 262, 310, 350, 382, 406, 422, 430\}$
479	$\{38, 118, 190, 254, 310, 358, 398, 430, 454, 470, 478\}$
503	$\{62, 142, 214, 278, 334, 382, 422, 454, 478, 494, 502\}$
743	$\{14, 118, 214, 302, 382, 454, 518, 574, 622, 662, 694, 718, 734, 742\}$
887	$\{46, 158, 262, 358, 446, 526, 598, 662, 718, 766, 806, 838, 862, 878, 886\}$

TABLE 5. $n(X_p)$ for some prime numbers $p \equiv 23 \pmod{24}$

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