THREE AMALGAMS WITH REMARKABLE NORMAL SUBGROUP STRUCTURES

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Abstract. We construct three groups Λ_1, Λ_2, Λ_3, which can all be decomposed as amalgamated products $F_9 *_{F_{81}} F_9$ and have very few normal subgroups of finite or infinite index. Concretely, Λ_1 is a simple group, Λ_2 is not simple but has no non-trivial normal subgroup of infinite index, and Λ_3 is not simple but has no proper subgroup of finite index.

1. Introduction

Motivated by expected analogies between cocompact lattices in products of automorphism groups of regular trees and cocompact lattices in higher rank semisimple Lie groups, Burger and Mozes discovered in their study of groups acting on products of trees the first examples of finitely presented torsion-free simple groups [5, 7]. These groups are moreover amalgamated products of finitely generated non-abelian free groups, thus answering Neumann's question [10] on the existence of simple amalgams of free groups. One crucial step in the construction of Burger-Mozes is a deep theorem, which states that certain cocompact lattices in the product of automorphism groups of locally finite trees $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ cannot have non-trivial normal subgroups of infinite index. Applying this theorem to a cocompact lattice which contains as a subgroup a non-residually finite group constructed by Wise in [14], we give an example of a finitely presented torsion-free simple group Λ_1 of the form $F_9 *_{F_{81}} F_9$, where F_k denotes the free group of rank k. See [12] for a list of 32 other finitely presented torsion-free simple groups emerging from the same method. Note that the simple groups of Burger-Mozes are also explicitly given in principle, but not very manageable in practice, because of their extremely long finite presentations. In addition to the simple group Λ_1, we construct two other groups Λ_2 and Λ_3, also having amalgam decompositions $F_9 *_{F_{81}} F_9$. They are not simple, but Λ_2 is virtually simple and Λ_3 has no non-trivial finite quotients. An amalgam $F_3 *_{F_{13}} F_3$ without proper subgroups of finite index has already been constructed by Bhattacharjee in [3], using different techniques. Our search for groups with the desired properties was made possible by several computer programs written in GAP [8]. See [11, Appendix B] for the program code used to construct the examples. We refer to [6], [7], [11] and [14] for detailed background on automorphism groups of trees, lattices in products of trees, and square complexes.

Date: September 22, 2006.

2000 *Mathematics Subject Classification.* Primary: 20E06. Secondary: 20E32.

Supported by the Swiss National Science Foundation, No. PP002–68627.

2. Definition of the groups Γ_i and Λ_i

Let always $i \in \{1,2,3\}$. Our groups Λ_i will be normal subgroups of index 4 of groups Γ_i defined by their finite presentations

$$\Gamma_i = \langle a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 \mid r_1, \ldots, r_{25} \rangle,$$

where the relators $r_1, \ldots, r_{25}$ (depending on i) are given in Table 1. Capital letters in this table indicate inverses, for example $r_1 = a_1 b_1 A_2 B_2 = a_1 b_1 a_2^{-1} b_2^{-1}$.

	Γ_1, Γ_2, Γ_3		Γ_1	Γ_2	Γ_3
r_1	$a_1b_1A_2B_2$	r_{13}	$a_1b_4a_2B_5$	$a_1b_4A_4b_5$	$a_1b_4a_1b_5$
r_2	$a_1b_2A_1B_1$	r_{14}	$a_1b_5A_5b_4$	$a_1b_5a_2B_5$	$a_1B_5a_2B_5$
r_3	$a_1b_3A_2B_3$	r_{15}	$a_1B_5a_3B_4$	$a_1B_5a_3B_4$	$a_1B_4A_4B_4$
r_4	$a_1B_3A_2b_2$	r_{16}	$a_1B_4a_3b_5$	$a_1B_4a_2b_4$	$a_2b_4a_2b_5$
r_5	$a_1B_1A_2b_3$	r_{17}	$a_2b_4A_2b_5$	$a_2b_5A_3b_4$	$a_2B_4A_3B_4$
r_6	$a_2b_2A_2B_1$	r_{18}	$a_2b_5a_4B_4$	$a_2B_4a_4B_5$	$a_3b_5a_4B_4$
r_7	$a_3b_1A_4B_2$	r_{19}	$a_3b_4a_4b_5$	$a_3b_4a_4b_4$	$a_3B_5A_5B_5$
r_8	$a_3b_2A_3B_1$	r_{20}	$a_3B_5a_4b_4$	$a_3b_5A_5b_5$	$a_3B_4a_4b_5$
r_9	$a_3b_3A_4B_3$	r_{21}	$a_4B_5A_5B_4$	$a_4b_5a_5b_5$	$a_4B_5a_5B_5$
r_{10}	$a_3B_3A_4b_2$	r_{22}	$a_5b_1A_5b_3$	$a_5b_1A_5b_3$	$a_5b_1a_5b_4$
r_{11}	$a_3B_1A_4b_3$	r_{23}	$a_5b_2A_5B_5$	$a_5b_2A_5B_1$	$a_5b_2A_5b_3$
r_{12}	$a_4b_2A_4B_1$	r_{24}	$a_5b_3A_5B_1$	$a_5b_3A_5B_4$	$a_5b_3A_5b_2$
		r_{25}	$a_5b_4A_5B_2$	$a_5b_4A_5B_2$	$a_5B_4a_5B_1$

Table 1. The 25 relators of Γ_1, Γ_2, Γ_3

Observe that the twelve relators $r_1, \ldots, r_{12}$ are the same for each group Γ_i. The reason for this will become clear in the proof of Theorem 1 in Section 3. To describe the geometric nature of Γ_i, we recall the following general construction which associates to a finite presentation of a group G its *standard 2-complex* X with fundamental group G: by definition, the one-skeleton of X has a single vertex x and an oriented loop for each generator of the given presentation of G. Furthermore, for each relator r, a 2-cell with boundary labelled by r is glued into this one-skeleton to get X. Then $G = \pi_1(X, x)$. By construction of the 25 relators of Γ_i, its associated standard 2-complex X_i is a finite square complex (all relators have length four, hence all 2-cells are squares) having the additional property that its universal cover $\tilde{X}_i$ is the affine building $\mathcal{T}_{10} \times \mathcal{T}_{10}$, the product of two 10-regular trees. Equivalently, this property requires that to each pair $(a,b) \in A \times B$, there is a uniquely determined pair $(\tilde{a}, \tilde{b}) \in A \times B$ such that $ab = \tilde{b}\tilde{a}$ in Γ_i, where $A := \{a_1, \ldots, a_5\}^{\pm 1}$ and $B := \{b_1, \ldots, b_5\}^{\pm 1}$. This can be easily verified for our three given examples. In the terminology of [7], X_i is a finite 1-vertex VH-T-square complex, and in the terminology of [11, 12], $\Gamma_i = \pi_1(X_i)$ is a $(10,10)$–group. The group of automorphisms $\mathrm{Aut}(\mathcal{T}_{10})$, equipped with the usual topology of simple convergence, is a locally compact group. Taking the product topology, Γ_i can be seen as a discrete subgroup of $\mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ with compact quotient, in other words as a cocompact lattice. A crucial role in deducing interesting results on the normal subgroup structure of Γ_i play the so-called local groups of Γ_i. The idea to define them is the following: take the projection of Γ_i to one factor of

$\mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ (say the projection pr_1 to the first factor) and fix any vertex x_h of $\mathcal{T}_{10}$. Then the elements in the closure $\overline{\mathrm{pr}_1(\Gamma_i)} < \mathrm{Aut}(\mathcal{T}_{10})$ stabilizing x_h, induce a finite permutation group $P_h^{(1)}(\Gamma_i) < S_{10}$ on the 10 neighbouring vertices of x_h in $\mathcal{T}_{10}$ (or more generally, for $k \in \mathbb{N}$, subgroups $P_h^{(k)}(\Gamma_i)$ of the symmetric group $S_{10 \cdot 9^{k-1}}$, taking the induced action on the k-sphere in $\mathcal{T}_{10}$ around x_h). The same procedure can be done with the second projection pr_2 to get local groups $P_v^{(k)}(\Gamma_i) < S_{10 \cdot 9^{k-1}}$. It is important to note that these local groups (more precisely, their generators in $S_{10 \cdot 9^{k-1}}$) can be directly computed, given the relators $r_1, \ldots, r_{25}$ of Table 1, see [7, Chapter 1] or [11, Section 1.4] for details. Here, we get for $k = 1$ the groups

$$\begin{aligned} P_h^{(1)}(\Gamma_1) = \langle &(7,8)(9,10), (1,2)(3,4), (1,2)(3,4)(7,8)(9,10), \\ &(1,8,4,5)(2,7,3,10), (1,9,4,8)(3,10,6,7)\rangle = A_{10}, \end{aligned}$$

$$\begin{aligned} P_h^{(1)}(\Gamma_2) = \langle &(7,8)(9,10), (1,2)(3,4), (1,2)(3,4)(7,8)(9,10), \\ &(1,8,4,9)(2,10,7,3), (1,9,8,6,4)(2,7,5,3,10)\rangle = A_{10}, \end{aligned}$$

$$\begin{aligned} P_h^{(1)}(\Gamma_3) = \langle &(5,6)(7,8)(9,10), (1,2)(3,4), (1,2)(3,4)(7,8)(9,10), \\ &(1,4,8,9,2,3,7,10)(5,6), (1,9,2,10)(3,5,7)(4,6,8)\rangle, \end{aligned}$$

$$\begin{aligned} P_v^{(1)}(\Gamma_1) = \langle &(1,2)(4,6,7,5)(8,10,9), (1,2,3)(4,5,7,6)(9,10), (1,2)(4,5,7,6)(8,10,9), \\ &(1,2,3)(4,6,7,5)(9,10), (1,3,10,8)(2,4,6,9,7,5)\rangle = A_{10}, \end{aligned}$$

$$\begin{aligned} P_v^{(1)}(\Gamma_2) = \langle &(1,2)(4,6)(8,10,9), (1,2,3)(5,7)(9,10), (1,2)(4,6,5,7)(8,10,9), \\ &(1,2,3)(4,6,5,7)(9,10), (1,2,4,3,10,9,7,8)(5,6)\rangle = A_{10}, \end{aligned}$$

$$\begin{aligned} P_v^{(1)}(\Gamma_3) = \langle &(1,2)(4,7,5,6)(8,10,9), (1,2,3)(4,7,5,6)(9,10), (1,2)(4,5,6,7)(8,10,9), \\ &(1,2,3)(4,5,6,7)(9,10), (1,7)(2,8)(3,9)(4,10)(5,6)\rangle = S_{10}. \end{aligned}$$

The transitivity of the permutation groups given above will be important in the proof of Theorem 1. Recall that a group $G < S_{10}$ is *transitive* if for any pair $m, n \in \{1, \ldots, 10\}$ there exists a $g \in G$ such that $g(m) = n$. Moreover, G is called 2-*transitive* if for any $m_1, m_2, n_1, n_2 \in \{1, \ldots, 10\}$ with $m_1 \neq m_2$ and $n_1 \neq n_2$ there is an element $g \in G$ such that $g(m_1) = n_1$ and $g(m_2) = n_2$. Note that the group $P_h^{(1)}(\Gamma_3)$ is a transitive (but not 2-transitive) subgroup of S_{10} of order 3840, whereas the alternating group A_{10} and the symmetric group S_{10} are obviously 2-transitive.

We define now Λ_i to be the kernel of the surjective homomorphism

$$\begin{aligned} \Gamma_i &\to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ a_1, \ldots, a_5 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\ b_1, \ldots, b_5 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}), \end{aligned}$$

where Γ_i is given by its finite presentation described above. Each group Λ_i can be decomposed in two ways as amalgamated products $F_9 *_{F_{81}} F_9$, such that F_{81} has index 10 in both factors F_9. More precisely, this means that for any $i \in \{1, 2, 3\}$ there exist injective homomorphisms $j_1, j_3 : F_{81} \to F_9 \cong \langle s_1, \ldots, s_9\rangle$ and $j_2, j_4 : F_{81} \to F_9 \cong \langle t_1, \ldots, t_9\rangle$ such that

$$[F_9 : j_1(F_{81})] = [F_9 : j_2(F_{81})] = [F_9 : j_3(F_{81})] = [F_9 : j_4(F_{81})] = 10$$

and

$$\begin{aligned}\Lambda_i &\cong \langle s_1, \dots, s_9, t_1, \dots, t_9 \mid j_1(u_1) = j_2(u_1), \dots, j_1(u_{81}) = j_2(u_{81}) \rangle \\ &\cong \langle s_1, \dots, s_9, t_1, \dots, t_9 \mid j_3(u_1) = j_4(u_1), \dots, j_3(u_{81}) = j_4(u_{81}) \rangle,\end{aligned}$$

where $\{u_1, \dots, u_{81}\}$ are the free generators of F_{81}. This is a direct consequence of a result of Wise (see [14, Theorem I.1.18]), describing each of the two decompositions of certain square complex groups Γ as a fundamental group of a finite graph of finitely generated free groups (in the language of the Bass-Serre theory). If the local groups of Γ are "sufficiently transitive" (which always happens in our examples), the two finite graphs corresponding to Λ_i in Wise's construction each consist of two vertices and a single geometric edge joining them. Therefore we get amalgams of finitely generated free groups. It is well-known that amalgams of free groups are always torsion-free, since every element of finite order in an amalgam is conjugate to an element of finite order in one of the two factors (see for example [9, Theorem IV.2.7]). Note that following Wise's proof of [14, Theorem I.1.18], it is not difficult (but quite laborious by hand) to give explicit descriptions of the injective homomorphisms $F_{81} \to F_9$ in the amalgam decompositions of Λ_i.

Alternatively, the decompositions $\Lambda_i \cong F_9 *_{F_{81}} F_9$ are consequences of the Bass-Serre theory. For example, using the local groups, one first checks that the quotient of the action of Λ_i on the first tree $\mathcal{T}_{10}$ (via the projection pr_1) is an edge. Then the identification (see [7, Chapter 1])

$$F_5 \cong \langle b_1, \dots, b_5 \rangle \cong \{\gamma \in \Gamma_i : \mathrm{pr}_1(\gamma)(x_h) = x_h\}$$

shows that the vertex stabilizers of this action are

$$\begin{aligned}F_9 &\cong \langle b_1 b_2^{-1}, b_1 b_3^{-1}, b_1 b_4^{-1}, b_1 b_5^{-1}, b_1 b_5, b_1 b_4, b_1 b_3, b_1 b_2, b_1^2 \rangle \\ &\cong \{\gamma \in \Lambda_i : \mathrm{pr}_1(\gamma)(x_h) = x_h\}.\end{aligned}$$

3. Results and Proofs

In the following theorem, we discuss the normal subgroups of Λ_i.

Theorem 1. *Let Λ_1, Λ_2, Λ_3 be the groups defined in Section 2. Then*

(1) Λ_1 *is simple.*
(2) *Every non-trivial normal subgroup of Λ_2 has finite index, but Λ_2 is not simple.*
(3) Λ_3 *has no proper subgroups of finite index, but is not simple.*

Proof. Let W be the group with finite presentation

$$\langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid r_1, \dots, r_{12} \rangle,$$

where the relators $r_1, \dots, r_{12}$ are again taken from Table 1. Wise showed in [14, Main Theorem 5.5], that the non-trivial element $w := a_2 a_1^{-1} a_3 a_4^{-1} \in W$ is contained in each finite index subgroup of W. In particular, W is non-residually finite. Moreover, $W < \mathrm{Aut}(\mathcal{T}_8) \times \mathrm{Aut}(\mathcal{T}_6)$ is the fundamental group of a 1-vertex VH-T square complex which embeds into the square complex X_i associated to Γ_i ($i = 1, 2, 3$), inducing an injection on the level of fundamental groups, i.e. $W < \Gamma_i = \pi_1(X_i)$ (the fact that we get an injection can be deduced from the non-positive curvature of the product of trees $\mathcal{T}_{10} \times \mathcal{T}_{10}$, see [4, Proposition II.4.14(1)]). Hence we have

$$1 \neq w \in \bigcap_{N \overset{\text{f.i.}}{\lhd} W} N < \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma_i} N = \bigcap_{N \overset{\text{f.i.}}{<} \Gamma_i} N \lhd \Gamma_i,$$

where "f.i." stands for "finite index". In particular, Γ_i (and hence its finite index subgroup Λ_i) is non-residually finite. Observe that $w \in \Lambda_i \lhd \Gamma_i$. One important point in the construction of Γ_i is to guarantee that the normal closure of w in Γ_i, denoted by $\langle\langle w \rangle\rangle_{\Gamma_i}$, has finite index in Λ_i. (Note that however $[W : \langle\langle w \rangle\rangle_W] = \infty$.) This already implies that $\langle\langle w \rangle\rangle_{\Gamma_i}$ has no proper subgroups of finite index. Indeed, assume that $M < \langle\langle w \rangle\rangle_{\Gamma_i}$ is a subgroup of finite index. Then

$$\bigcap_{N \overset{\text{f.i.}}{<} \Gamma_i} N < M \overset{\text{f.i.}}{<} \langle\langle w \rangle\rangle_{\Gamma_i} \overset{\text{f.i.}}{<} \Lambda_i \overset{\text{f.i.}}{<} \Gamma_i.$$

Using

$$\langle\langle w \rangle\rangle_{\Gamma_i} < \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma_i} N = \bigcap_{N \overset{\text{f.i.}}{<} \Gamma_i} N,$$

we get

$$M = \langle\langle w \rangle\rangle_{\Gamma_i} = \bigcap_{N \overset{\text{f.i.}}{<} \Gamma_i} N.$$

We proceed now separately for the three groups Λ_1, Λ_2 and Λ_3.

(1) We have $\langle\langle w \rangle\rangle_{\Gamma_1} = \Lambda_1$. This can be checked by hand, or more easily, using a computer algebra system like GAP [8], which shows that adding the relator w to the presentation of Γ_1 gives the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of order 4. It remains to prove that Λ_1 has no non-trivial normal subgroups of infinite index. But this follows directly from the normal subgroup theorem of Burger-Mozes [7, Theorem 4.1, Corollary 5.4] applied to the "irreducible" cocompact lattice $\Gamma_1 < \mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ with local groups $P_h^{(1)}(\Gamma_1) \cong P_v^{(1)}(\Gamma_1) = A_{10}$, and applied to its finite index subgroup $\Lambda_1 < \Gamma_1$.

(2) For the second group, we compute $[\Lambda_2 : \langle\langle w \rangle\rangle_{\Gamma_2}] = 2$, thus Λ_2 is not simple. By exactly the same argument as in part (1), every non-trivial normal subgroup of Γ_2 (and of Λ_2, respectively) has finite index. Observe that $\langle\langle w \rangle\rangle_{\Gamma_2}$ is a simple group with amalgam decomposition $F_{17} *_{F_{161}} F_{17}$. In particular, Γ_2 and Λ_2 are virtually simple groups.

(3) As in part (1), $\langle\langle w \rangle\rangle_{\Gamma_3} = \Lambda_3$ proves that Λ_3 has no proper subgroup of finite index. However, in contrast to what happens in part (1) and (2), the local group $P_h^{(1)}(\Gamma_3)$ is transitive, but not 2-transitive. Therefore, the normal subgroup theorem of Burger-Mozes cannot be applied here. Indeed, Λ_3 is not simple, since $1 \neq \langle\langle a_5^4 \rangle\rangle_{\Lambda_3} \neq \Lambda_3$. This comes from the fact that a_5^4 acts trivially on the second factor of $\mathcal{T}_{10} \times \mathcal{T}_{10}$. In other words, $a_5^4 \in \ker(\mathrm{pr}_2) \lhd \Gamma_3$. To see this, let

$$A' := \{(a_1 a_2^{-1})^2, (a_2^{-1} a_1)^2, (a_3 a_4^{-1})^2, (a_4^{-1} a_3)^2, a_5^4\}^{\pm 1}$$

and check that for all $a' \in A'$ and $b \in B = \{b_1, \ldots, b_5\}^{\pm 1}$, we have $b^{-1} a' b \in A'$. This in fact implies that $A' \subset \ker(\mathrm{pr}_2)$. Note that no element of Γ_3 acts trivially on the *first* factor of $\mathcal{T}_{10} \times \mathcal{T}_{10}$ (by [6, Proposition 3.1.2, 1)] and [6, Proposition 3.3.2]). As a consequence, Λ_3 has two decompositions $F_9 *_{F_{81}} F_9$, where one amalgam is effective and the other one is not effective. $\square$

We conclude by giving two remarks:

Remark 2. Recall that a group G is called *SQ-universal* if every countable group can be embedded in a quotient of G. It is mentioned in [1, Chapter 9.15] that Ilya Rips can prove any amalgamated product $A *_C B$ to be SQ-universal, provided that $B \neq C$ and the number of double cosets $|C \backslash A / C|$ is at least 3 (if C is seen as usual as a subgroup of A and B via the two injections $j_1 : C \to A$ and $j_2 : C \to B$ in the amalgam), but there is no published proof as far as we know. If Rips' statement is true, we could apply it to exactly one decomposition $F_9 *_{F_{81}} F_9$ of Λ_3 (to the effective one), where $|F_{81} \backslash F_9 / F_{81}| = 3$. Note however that in the second decomposition of Λ_3 (where the corresponding local group $P_v^{(1)}(\Gamma_3)$ is S_{10}) and in both decompositions of Λ_1 and Λ_2, we always have $|F_{81} \backslash F_9 / F_{81}| = 2$, since their local actions on $\mathcal{T}_{10}$ are 2-transitive.

Remark 3. By construction, the three groups Λ_1, Λ_2, Λ_3 are non-residually finite. As a contrast, if one takes a *double* $F_9 *_{F_{81}} F_9$ (i.e. an amalgam where the two injections $j_1, j_2 : F_{81} \to F_9$ are identical), such that F_{81} has finite index in both factors F_9 (consequently index $10 = (81-1)/(9-1)$), then one directly gets a surjective homomorphism $F_9 *_{F_{81}} F_9 \to F_9$ (the obvious folding map), and moreover $F_9 *_{F_{81}} F_9$ contains by [2, Theorem 1.4] a subgroup of finite index which is a direct product of two non-abelian free groups of finite rank. In particular, such a double $F_9 *_{F_{81}} F_9$ is SQ-universal and residually finite. The residual finiteness also follows from [13].

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