Examples of Square Complexes

Diego Rattaggi

August 20, 2003

(preliminary version, available on
`www.math.ethz.ch/~rattaggi/publ`)

Contents

0 Introduction

In [14], Marc Burger and Shahar Mozes have constructed the first examples of finitely presented torsion-free simple groups. These groups are cocompact lattices in a product $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ of automorphism groups of regular trees, and they are moreover biautomatic, of finite cohomological dimension and decomposable as amalgamated free products of finitely generated free groups. One important step in their construction is a theorem (an analogue of Margulis' normal subgroup theorem in the context of Lie groups), saying that for a certain class of lattices $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, every non-trivial normal subgroup of Γ has finite index. In order to apply this theorem, strong assumptions on the local transitivity properties of both projections of Γ have to be made. The aim of this paper is to give many examples of lattices $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ whose projections satisfy various transitivity conditions, for example those assumed in the normal subgroup theorem just mentioned. For the most part, we will restrict to groups Γ acting freely and vertex-transitively on the simply connected 2-dimensional cellular complex $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. In this case, Γ is the fundamental group of a square complex with $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$ as its universal covering space and having only one vertex.

More precise definitions of the involved groups and cell complexes (respectively their relation) are given in Section 1. In that section, we also fix our notations and recall some important notions and tools developed in [14], [15], [16], useful in the study of those lattices, like for example "local groups" or "irreducibility".

Section 2 is mainly concerned with explicit constructions of irreducible cocompact lattices $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, whose two projections act locally like the alternating group A_6 and locally primitively, respectively. In particular, we try to analyze in more detail two examples (Example 1 and Example 3), which seem to be quite similar at first sight, but are very different regarding their normal subgroup structures. However, several central questions remain open for the moment and some conjectures are formulated. We believe for instance that Example 1 immediately leads to a simple group, also sharing all other properties of the remarkable groups constructed in [14], but which is in contrast to those groups much more easy to describe. For example, besides the natural presentation as fundamental group of the corresponding square complex, we give explicit amalgam decompositions.

In Section 3, we first apply several results of [14], [15], [16] to construct a non-residually finite lattice in $\mathrm{Aut}(\mathcal{T}_4) \times \mathrm{Aut}(\mathcal{T}_{12})$ (Example 8). Then we embed it into finitely presented torsion-free cocompact lattices in $\mathrm{Aut}(\mathcal{T}_6) \times \mathrm{Aut}(\mathcal{T}_{16})$ and $\mathrm{Aut}(\mathcal{T}_8) \times \mathrm{Aut}(\mathcal{T}_{14})$ respectively, which are virtually simple (Example 9, Example 10 and Example 11). Note that the "smallest" such group constructed by Burger and Mozes is a lattice in $\mathrm{Aut}(\mathcal{T}_{218}) \times \mathrm{Aut}(\mathcal{T}_{350})$. An embedding of a known non-residually finite group (Example 13), presented by Daniel T. Wise in [69], finally leads to the construction of a simple cocompact lattice in $\mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ of the form $F_9 *_{F_{81}} F_9$. Using similar techniques, we construct in this section many more such simple groups (Table 8), but also a non-simple group in $\mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ without proper subgroups of finite index (Example 18) and a non-residually finite, not virtually torsion-free infinite quotient of a cocompact lattice in $\mathrm{Aut}(\mathcal{T}_8) \times \mathrm{Aut}(\mathcal{T}_8)$ (Example 19).

In Section 4, we examine some possible connections between (ir)reducibility of Γ, (transitivity) properties of the local permutation groups and the size of $\Gamma/[\Gamma, \Gamma]$. We illustrate for instance, that it is not possible to give a general criterion for irreducibility just in terms of the local groups P_h and P_v. Moreover, some irreducible examples with non-trivial quasi-center (in one or both projections) are given.

In Section 5, we give examples of "quaternion lattices" in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$, where p and l are distinct odd prime numbers. We realize these groups in several ways, in particular as linear groups or as group of units (modulo center) of quaternions over a certain ring, and study finite quotients $\mathrm{PGL}_2(q)$ and $\mathrm{PSL}_2(q)$. We try to give very detailed constructions and proofs in this section. The standard construction for $p, l \equiv 1 \pmod 4$ is taken from [53] (cf. [43]), but we extend it here and give some conjectures in particular concerning their abelianization. This theory has several applications. For example, by proving that the groups are "commutative transitive", we give a simple criterion for the existence of "anti-tori". Moreover, we prove certain integer quaternions to have non-trivial relations and discuss the existence of free anti-tori and free subgroups in $\mathrm{SO}_3(\mathbb{Q})$.

In Section 6, we prove that our groups always have subgroups isomorphic to $\mathbb{Z}^2$. This is related to the existence of periodic tilings of the plane.

In Section 7, we construct a very small 4-vertex square complex, whose fundamental group is a possible candidate for a finitely presented torsion-free simple group. More 4-vertex examples are given, demonstrating that the local groups can behave quite differently than in the 1-vertex case.

In Appendix A, we give some supplements to Section 2: more examples, explicit amalgam decompositions, and some detailed proofs.

We refer to Appendix C for some complete lists in "small dimensions", which already confirm the great variety and richness of 1-vertex square complexes.

Most examples of this paper have been found by means of several computer programs written in GAP ([28]). The main programs are listed and described in Appendix D.

Appendix E is reserved to miscellaneous subjects, including a survey of the history of finitely presented infinite simple groups and amalgams of free groups.

1 Preliminaries, notations and definitions

Throughout this article, the main object of our study will be a special class of square complexes which we want to define now. We always assume that $m, n \in \mathbb{N}$.

Definition. A $(2m, 2n)$-*complex* X is a finite 2-dimensional cell complex satisfying the following properties: Its 1-skeleton $X^{(1)}$ is a connected graph $(X^{(0)}, E)$ with vertex set $X^{(0)} = \{x\}$ consisting of only one single vertex x and edge set $E = E_h \sqcup E_v$, decomposed into m geometric "horizontal" loops $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$ and n geometric "vertical" loops $b_1^{\pm 1}, \ldots, b_n^{\pm 1}$. Following Serre's terminology for graphs (see [66, Chapter 2.1]), this means that

$$E_h = \{a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}\},\ \overline{a_i} = a_i^{-1},\ o(a_i) = t(a_i) = x, \quad i = 1, \ldots, m$$
$$E_v = \{b_1, \ldots, b_n, b_n^{-1}, \ldots, b_1^{-1}\},\ \overline{b_j} = b_j^{-1},\ o(b_j) = t(b_j) = x, \quad j = 1, \ldots, n.$$

The desired cell complex X is now constructed by attaching mn geometric squares to $X^{(1)}$, where the four paths in the boundary of each geometric square are identified alternately with edges in E_h and E_v. This has to be done in such a way that the link $Lk(x)$ of the vertex x becomes a complete bipartite graph $K_{2m,2n}$ (where the bipartite structure then is induced by the decomposition of E as $E_h \sqcup E_v$). Equivalently, we require that the universal covering space $\tilde{X}$ of X is a product of two regular trees $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$, where $\mathcal{T}_\ell$ denotes the (infinite) ℓ-regular tree.

Example. See Figure 1 for an example of a $(2, 4)$-complex X.

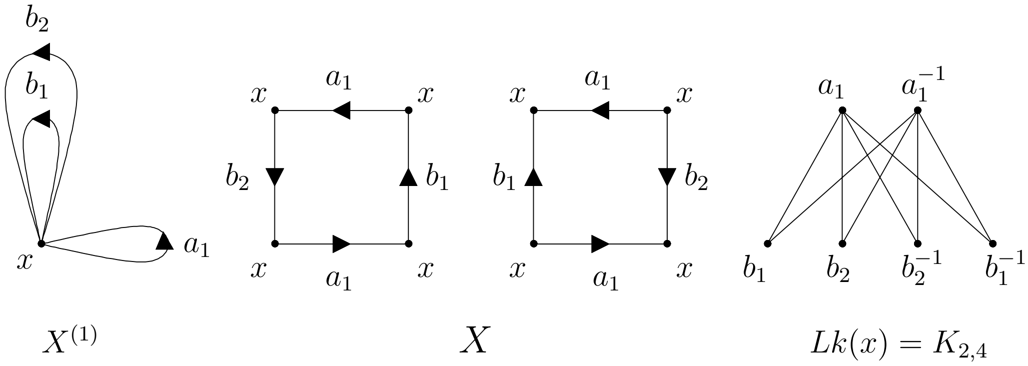


Figure 1: A $(2, 4)$-complex X

By construction, the fundamental group $\Gamma := \pi_1(X, x) < \mathrm{Aut}(\mathcal{T}_{2m} \times \mathcal{T}_{2n})$ is a torsion-free cocompact lattice, acting freely and vertex-transitively on $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. The decomposition of E guarantees that $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n}) < \mathrm{Aut}(\mathcal{T}_{2m} \times \mathcal{T}_{2n})$. Such a group Γ will be called a $(2m, 2n)$-*group*.

A finite presentation of Γ can be directly read off from X:

$$\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m, n)\rangle,$$

where the mn relators in $R(m, n)$ are words of length 4 in the generators a_i, b_j and their inverses, coming from the mn geometric squares in X. In our example (Figure 1):

$$R(1, 2) = \{a_1 b_1 a_1 b_2,\ a_1 b_2^{-1} a_1 b_1^{-1}\},$$

i.e.

$$\Gamma = \langle a_1, b_1, b_2 \mid a_1 b_1 a_1 b_2, a_1 b_2^{-1} a_1 b_1^{-1}\rangle.$$

Note that there are several different possibilities to describe a geometric square, e.g. in the example above we have equalities (as geometric squares and as relators)

$$a_1b_1a_1b_2 = a_1b_2a_1b_1 = a_1^{-1}b_2^{-1}a_1^{-1}b_1^{-1} = a_1^{-1}b_1^{-1}a_1^{-1}b_2^{-1} =$$

$$b_1a_1b_2a_1 = b_2a_1b_1a_1 = b_1^{-1}a_1^{-1}b_2^{-1}a_1^{-1} = b_2^{-1}a_1^{-1}b_1^{-1}a_1^{-1}.$$

Any such expression represents the same geometric square and all constructions involving geometric squares (like e.g. the group Γ up to isomorphism) will be independent of the choice of representatives.

Given a $(2m, 2n)$-group Γ, we can define a surjective homomorphism of groups

$$\begin{aligned} &\Gamma \to \mathbb{Z}_2^2 \\ &a_i \mapsto (1,0), \quad i = 1, \ldots, m \\ &b_j \mapsto (0,1), \quad j = 1, \ldots, n \end{aligned}$$

with $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} \cong \{0,1\}$ written additively. Obviously, the kernel of this homomorphism is a normal subgroup of Γ of index 4. We denote this group by Γ_0. It is the fundamental group of a corresponding square complex X_0 with 4 vertices, a 4-fold regular covering of X.

For more details and a more general definition of square complexes, see [16] (see also [69]).

Definition. Since $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, we have two canonical projections, the group homomorphisms $\mathrm{pr}_1 : \Gamma \to \mathrm{Aut}(\mathcal{T}_{2m})$ and $\mathrm{pr}_2 : \Gamma \to \mathrm{Aut}(\mathcal{T}_{2n})$. We define $H_i := \overline{\mathrm{pr}_i(\Gamma)}$, $i = 1, 2$, where the closure is taken with respect to the topology of pointwise convergence. With this topology, $\mathrm{Aut}(\mathcal{T}_\ell)$, $\ell \geq 3$, is a locally compact, totally disconnected, uncountable topological group (see Appendix E.1). Let

$$\mathrm{QZ}(H_i) := \{h \in H_i : \text{centralizer } Z_{H_i}(h) \text{ is open in } H_i\}$$

be the *quasi-center* of H_i (see [15] for an introduction to this group). Finally, we put

$$\Lambda_1 := \mathrm{pr}_1(\Gamma \cap (H_1 \times \{1\})) = \mathrm{pr}_1(\Gamma \cap (\mathrm{Aut}(\mathcal{T}_{2m}) \times \{1\})) < \mathrm{Aut}(\mathcal{T}_{2m})$$

and

$$\Lambda_2 := \mathrm{pr}_2(\Gamma \cap (\{1\} \times H_2)) = \mathrm{pr}_2(\Gamma \cap (\{1\} \times \mathrm{Aut}(\mathcal{T}_{2n}))) < \mathrm{Aut}(\mathcal{T}_{2n}).$$

Observe that

$$\Lambda_i = \mathrm{pr}_i(\ker(\mathrm{pr}_{3-i})) \cong \ker(\mathrm{pr}_{3-i}) \lhd \Gamma$$

and note that $\Lambda_i \lhd \mathrm{QZ}(H_i)$, since every discrete normal subgroup of H_i is contained in $\mathrm{QZ}(H_i)$, as explained in [15]. In particular, we conclude that $\mathrm{QZ}(H_i) = 1$ implies an isomorphism $\Gamma \cong \mathrm{pr}_{3-i}(\Gamma)$ and in this case we can see Γ as a subgroup of $\mathrm{Aut}(\mathcal{T}_{2m})$, if $i = 2$, or as a subgroup of $\mathrm{Aut}(\mathcal{T}_{2n})$, if $i = 1$.

We now turn to the definition of the "local groups" P_h and P_v, which will play a major role in the construction of interesting groups Γ. Let $E_v^{(k)}$ and $E_h^{(k)}$ be the set of vertical respectively horizontal reduced paths (i.e. without backtracking) of combinatorial length $k \in \mathbb{N}$ in $X^{(1)}$. In particular, $E_v^{(1)} = E_v$, $E_h^{(1)} = E_h$,

$$|E_v^{(k)}| = 2n \cdot (2n-1)^{k-1} \text{ and } |E_h^{(k)}| = 2m \cdot (2m-1)^{k-1}.$$

There is a family of homomorphisms from the free group F_m of rank m generated by $\{a_1, \ldots, a_m\}$ to the symmetric group of the set $E_v^{(k)}$

$$\rho_h^{(k)} : F_m = \langle a_1, \ldots, a_m \rangle \to \mathrm{Sym}(E_v^{(k)}) \cong S_{2n\cdot(2n-1)^{k-1}}$$

and a family of homomorphisms

$$\rho_v^{(k)} : F_n = \langle b_1, \dots, b_n \rangle \to \mathrm{Sym}(E_h^{(k)}) \cong S_{2m\cdot(2m-1)^{k-1}}.$$

We denote their images by

$$P_v^{(k)} := \mathrm{im}(\rho_h^{(k)}) = \langle \rho_h^{(k)}(a_1), \dots, \rho_h^{(k)}(a_m) \rangle \text{ and } P_h^{(k)} := \mathrm{im}(\rho_v^{(k)}) = \langle \rho_v^{(k)}(b_1), \dots, \rho_v^{(k)}(b_n) \rangle.$$

For $k = 1$, we omit the superscript "(1)" and simply write

$$\rho_h : \langle a_1, \dots, a_m \rangle \twoheadrightarrow \langle \rho_h(a_1), \dots, \rho_h(a_m) \rangle = P_v < \mathrm{Sym}(E_v) = \mathrm{Sym}(\{b_1, \dots, b_n, b_n^{-1}, \dots, b_1^{-1}\}) \cong S_{2n},$$

where for the last isomorphism we always use the explicit identification

$$\begin{aligned} E_v &\cong \{1, \dots, 2n\} \\ b_j &\leftrightarrow j, \\ b_j^{-1} &\leftrightarrow 2n + 1 - j, \end{aligned}$$

$j = 1, \dots, n$, and

$$\rho_v : \langle b_1, \dots, b_n \rangle \twoheadrightarrow \langle \rho_v(b_1), \dots, \rho_v(b_n) \rangle = P_h < \mathrm{Sym}(E_h) = \mathrm{Sym}(\{a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1}\}) \cong S_{2m},$$

via the identification

$$\begin{aligned} E_h &\cong \{1, \dots, 2m\} \\ a_i &\leftrightarrow i, \\ a_i^{-1} &\leftrightarrow 2m + 1 - i, \end{aligned}$$

for $i = 1, \dots, m$. The two homomorphisms ρ_h and ρ_v are defined as follows: each relator $aba'b'$ in $R(m, n)$ gives

$$\begin{aligned} \rho_h(a)(b'^{-1}) &= b \\ \rho_h(a')(b^{-1}) &= b' \\ \rho_v(b)(a^{-1}) &= a' \\ \rho_v(b')(a'^{-1}) &= a, \end{aligned}$$

as indicated in Figure 2.

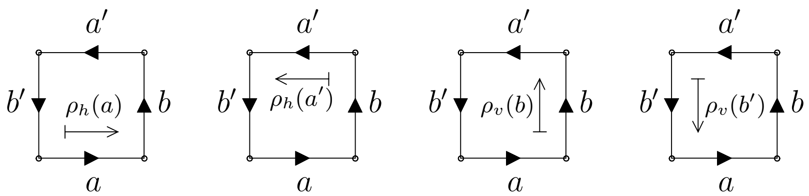


Figure 2: Visualizing the definition of ρ_h, ρ_v

By the link condition in X, these $4mn$ expressions indeed uniquely determine ρ_h and ρ_v. We obtain in our example in Figure 1 $\rho_h(a_1) = (1, 3)(2, 4)$, $\rho_v(b_1) = (1, 2)$, $\rho_v(b_2) = (1, 2)$, hence

$$P_v = \langle (1, 3)(2, 4) \rangle \cong \mathbb{Z}_2 < S_4 \text{ and } P_h = \langle (1, 2) \rangle = S_2.$$

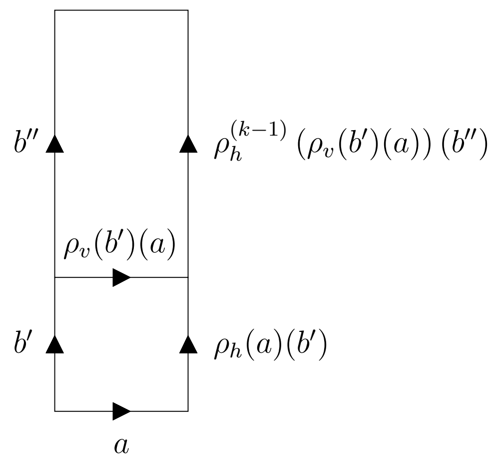


Figure 3: Inductive definition of $\rho_h^{(k)}$, $k \geq 2$

If $k \geq 2$, the homomorphisms $\rho_h^{(k)}$ and $\rho_v^{(k)}$ are defined in a similar way, see [16, Chapter 1]. We give an inductive definition of $\rho_h^{(k)}$: Let $a \in E_h$ and $b = b' \cdot b'' \in E_v^{(k)}$, where $b' \in E_v$, $b'' \in E_v^{(k-1)}$. Then

$$\rho_h^{(k)}(a)(b) := \rho_h(a)(b') \cdot \rho_h^{(k-1)}\left(\rho_v(b')(a)\right)(b''),$$

see Figure 3 for an illustration. The homomorphism $\rho_v^{(k)}$ can be defined analogously.

Starting with X, the finite permutation groups $P_v^{(k)}$ and $P_h^{(k)}$ can be effectively computed. They describe the local actions of the projections of Γ on k-spheres in $\mathcal{T}_{2n}$ and $\mathcal{T}_{2m}$ respectively. More precisely, let x_v be any vertex in $\mathcal{T}_{2n}$ and let $S(x_v, k)$ be the k-sphere (i.e. the set of vertices in $\mathcal{T}_{2n}$ of combinatorial distance k from x_v), then the groups

$$P_v^{(k)} < \mathrm{Sym}(E_v^{(k)}) \text{ and } H_2(x_v)/H_2(S(x_v,k)) < \mathrm{Sym}(S(x_v,k))$$

are permutation isomorphic (see [16, Chapter 1]), where for $H < \mathrm{Aut}(\mathcal{T}_\ell)$ and S a subset of vertices in $\mathcal{T}_\ell$, we always write $H(S)$ to denote the *pointwise* stabilizer

$$\mathrm{Stab}_H(S) = \{h \in H : h(s) = s, \forall s \in S\}.$$

The analogous statement holds for $P_h^{(k)}$ and $H_1(x_h)/H_1(S(x_h,k))$, where x_h is any vertex in $\mathcal{T}_{2m}$.

For each $k \in \mathbb{N}$, there is a commutative diagram

$$\begin{tikzcd}
\langle a_1, \ldots, a_m \rangle \arrow[r, two heads, "\rho_h^{(k+1)}"] \arrow[dr, two heads, "\rho_h^{(k)}"'] & P_v^{(k+1)} < \mathrm{Sym}(E_v^{(k+1)}) \arrow[d, two heads, "p_k"] \\
 & P_v^{(k)} < \mathrm{Sym}(E_v^{(k)})
\end{tikzcd}$$

where p_k is the homomorphism restricting the action of $P_v^{(k+1)}$ on the $(k+1)$-sphere $S(x_v, k+1)$ to the k-sphere $S(x_v, k)$. In particular, $\left|P_v^{(k)}\right|$ divides $\left|P_v^{(k+1)}\right|$. Note that

$$\bigcap_{k \in \mathbb{N}} \ker \rho_h^{(k)} \cong \Lambda_1 \text{ and } \bigcap_{k \in \mathbb{N}} \ker \rho_v^{(k)} \cong \Lambda_2.$$

In general, a subgroup $H < \mathrm{Aut}(\mathcal{T}_\ell)$ is called *locally transitive (locally primitive, locally* 2-*transitive, ...)* if the stabilizer in H of every vertex x in $\mathcal{T}_\ell$ induces a transitive (primitive, 2-transitive, ...) permutation group on the ℓ neighbouring vertices of x. Finally, we call H *locally* ∞-*transitive*, if $H(x)$ acts transitively on $S(x,k)$ for each $k \geq 1$ and each vertex x in $\mathcal{T}_\ell$.

Because of the importance of the local groups P_h and P_v in our study of X, we will often call X a (P_h, P_v)*-complex* and the corresponding fundamental group Γ a (P_h, P_v)*-group*.

One of the main notions in the theory of lattices in semisimple Lie groups is irreducibility. In our case, we adopt the definition given in [16].

Definition. A $(2m, 2n)$-group Γ is called *reducible* if $\mathrm{pr}_1(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2m})$ is discrete. Otherwise, Γ is called *irreducible*. A $(2m, 2n)$-complex X is said to be (ir)reducible if and only if $\Gamma = \pi_1(X, x)$ is (ir)reducible.

Remarks. (1) Observe that a subgroup of $\mathrm{Aut}(\mathcal{T}_\ell)$ is discrete if and only if its vertex stabilizers are all finite (see Proposition 79 in Appendix E.1).

(2) It is shown in [16, Proposition 1.2] that $\mathrm{pr}_1(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2m})$ is discrete if and only if $\mathrm{pr}_2(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2n})$ is discrete.

(3) Note that $\mathrm{pr}_1(\Gamma)$ is never dense in $\mathrm{Aut}(\mathcal{T}_{2m})$, i.e. $H_1 \subsetneqq \mathrm{Aut}(\mathcal{T}_{2m})$, contrary to "irreducible" lattices in higher rank semisimple Lie groups.

(4) In terms of orders of the local groups $P_h^{(k)}$ and $P_v^{(k)}$, Γ is reducible if and only if the set $\{|P_h^{(k)}|\}_{k\in\mathbb{N}}$ is bounded, if and only if $\{|P_v^{(k)}|\}_{k\in\mathbb{N}}$ is bounded.

In geometric terms, X is reducible if and only if X admits a finite covering which is a product of two graphs (see [16, Chapter 1]). Therefore, a reducible group Γ is virtually a direct product of two free groups of finite rank, in particular Γ is residually finite, i.e. the intersection of finite index subgroups of Γ is trivial. As a consequence, a non-residually finite Γ has to be irreducible. In general, no algorithm is known to determine whether a given Γ is reducible or not. However, a useful sufficient criterion for irreducibility, based on the Thompson-Wielandt theorem (see e.g. [15, Theorem 2.1.1]), is presented in [16, Proposition 1.3]. We will strongly use the criteria (1) and (2) of the following proposition based on results in [15] and [16]. The third criterion (3) will only be used in Theorem 8, where (1) does not apply.

Proposition 1. *Let Γ be a $(2m, 2n)$-group.*

(1*a*) *Suppose that $P_h = A_{2m}$, $m \geq 3$. Then Γ is irreducible if and only if*

$$\left|P_h^{(2)}\right| = |A_{2m}| \left(\frac{|A_{2m}|}{2m}\right)^{2m} = \frac{(2m)!}{2}\left(\frac{(2m-1)!}{2}\right)^{2m}.$$

(1*b*) *Suppose that $P_v = A_{2n}$, $n \geq 3$. Then Γ is irreducible if and only if*

$$\left|P_v^{(2)}\right| = |A_{2n}| \left(\frac{|A_{2n}|}{2n}\right)^{2n} = \frac{(2n)!}{2}\left(\frac{(2n-1)!}{2}\right)^{2n}.$$

(2*a*) Γ *is reducible if and only if there is a number $k \in \mathbb{N}$ such that $|P_h^{(k+1)}| = |P_h^{(k)}|$.*

(2*b*) Γ *is reducible if and only if there is a number $k \in \mathbb{N}$ such that $|P_v^{(k+1)}| = |P_v^{(k)}|$.*

(3*a*) *Let $P_h < S_{2m}$ be transitive and suppose that for each $k \geq 1$ there exist reduced words $b \in \langle b_1, \ldots, b_n\rangle$ and $a \in \langle a_1, \ldots, a_m\rangle$ with $|a| = k$ such that $\rho_v^{(k)}(b)(a) = a$ and $\rho_v(\tilde{b})$ acts transitively on $E_h \setminus \{a''^{-1}\}$, where $\tilde{b} := \rho_h^{(|b|)}(a)(b)$ and $a = a' \cdot a''$ with $a' \in E_h^{(k-1)}$, $a'' \in E_h$ (see Figure 4). Then $\mathrm{pr}_1(\Gamma)$ is locally ∞-transitive, in particular Γ is irreducible.*

(3*b*) *Let $P_v < S_{2n}$ be transitive and suppose that for each $k \geq 1$ there exist reduced words $a \in \langle a_1, \ldots, a_m\rangle$ and $b \in \langle b_1, \ldots, b_n\rangle$ with $|b| = k$ such that $\rho_h^{(k)}(a)(b) = b$ and $\rho_h(\tilde{a})$ acts transitively on $E_v \setminus \{b''^{-1}\}$, where $\tilde{a} := \rho_v^{(|a|)}(b)(a)$ and $b = b' \cdot b''$ with $b' \in E_v^{(k-1)}$, $b'' \in E_v$. Then $\mathrm{pr}_2(\Gamma)$ is locally ∞-transitive, in particular Γ is irreducible.*

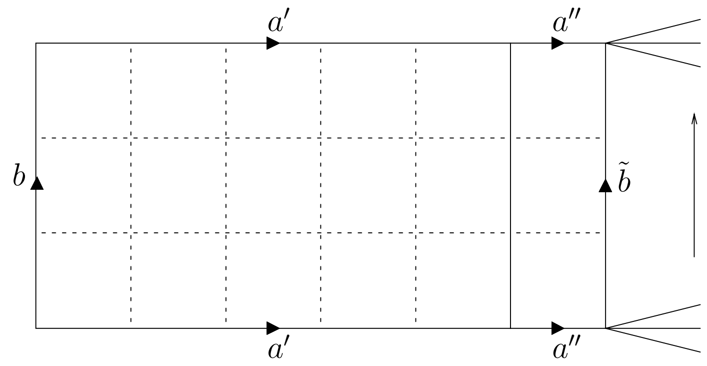


Figure 4: Notations in Proposition 1(3a)

Proof. We prove (1a), (2a) and (3a). The proofs of (1b), (2b) and (3b) are analogous.

(1a) The statement follows directly from [15, Proposition 3.3.1].

(2a) Obviously, $|P_h^{(k+1)}| = |P_h^{(k)}|$ for some $k \in \mathbb{N}$ is a necessary condition. We want to prove now, that it is also sufficient for the reducibility of Γ. It is enough to show $|P_h^{(k+2)}| = |P_h^{(k+1)}|$. First observe that for all vertices $x_h \in \mathcal{T}_{2m}$ we have

$$H_1(S(x_h, k+1)) = H_1(S(x_h, k)) \qquad (*)$$

since

$$1 = \left| P_h^{(k+1)} / P_h^{(k)} \right| = \left| H_1(S(x_h, k)) / H_1(S(x_h, k+1)) \right| .$$

Assume now that $|P_h^{(k+2)}| > |P_h^{(k+1)}|$. It follows that there is a $g \in H_1(S(x_h, k+1)) \setminus H_1(S(x_h, k+2))$. But then, for at least one neighbouring vertex y_h of x_h,

$$g \in H_1(S(y_h, k)) \setminus H_1(S(y_h, k+1)),$$

contradicting $(*)$.

Note that we construct in Example 57 an irreducible 4-vertex square complex such that $|P_h^{(2)}| = |P_h|$ for some vertex.

(3a) We have to show that $\mathrm{pr}_1(\Gamma)(x_h)$ acts transitively on $S(x_h, k)$ for each $k \geq 1$. This is done by induction on k using the identification (see [16, Chapter 1])

$$\langle b_1, \ldots, b_n \rangle \cong \{\gamma \in \Gamma : \mathrm{pr}_1(\gamma)(x_h) = x_h\}.$$

For $k = 1$, the statement is obvious since P_h is transitive. To prove the induction step $k \to k+1$, note that by induction hypothesis, $\mathrm{pr}_1(\Gamma)(x_h)$ acts transitively on $S(x_h, k)$, hence we have at most $2m-1$ orbits in $S(x_h, k+1)$. But now, the assumptions (in particular the transitivity of $\rho_v(\tilde{b})$) exactly guarantee that there is in fact only one orbit.

Since $P_h^{(k)}$ is transitive for each $k \geq 1$, the set $\{|P_h^{(k)}|\}_{k \in \mathbb{N}}$ is not bounded and therefore Γ is irreducible.

$\square$

Remark. Note that Proposition 1(1a) is false if $P_h = A_4$ (i.e. if $m = 2$), because there are irreducible (A_4, A_{10})-groups such that

$$|P_h^{(2)}| = 324 < |A_4| \left(\frac{|A_4|}{4} \right)^4 = 972$$

(cf. Appendix C.4).

It is shown in [69, Theorem I.1.18] that Γ splits in two ways as a fundamental group of a finite graph of finitely generated free groups (in the terminology of Bass-Serre theory). We are mainly interested in amalgamated free products of free groups, i.e. fundamental groups of edges of free groups.

Proposition 2. *Let Γ be a $(2m, 2n)$-group.*

(1a) *If $P_h < S_{2m}$ is a transitive permutation group, then Γ can be written as an amalgamated free product of finitely generated free groups:*

$$\Gamma \cong F_n *_{F_{1-2m+2mn}} F_{1-m+mn}.$$

We call it the vertical decomposition *of Γ.*

(1b) *If $P_v < S_{2n}$ is transitive, then we have a* horizontal decomposition

$$\Gamma \cong F_m *_{F_{1-2n+2mn}} F_{1-n+mn}.$$

Proof. The statements follow directly from [69, Theorem I.1.18] after subdividing the complex X vertically in the first case and horizontally in the second case. $\square$

Observe that for the indices in the two inclusions in the splitting in Proposition 2(1a) we have $[F_n : F_{1-2m+2mn}] = 2m$ and $[F_{1-m+mn} : F_{1-2m+2mn}] = 2$. The tree on which Γ naturally acts is $\mathcal{T}'_{2m}$, the first barycentric subdivision of $\mathcal{T}_{2m}$. Note that $F_n = \langle b_1, \ldots, b_n\rangle$. Further, the second factor F_{1-m+mn} is the fundamental group of a graph with m vertices (one for each geometric edge $a_i^{\pm 1}$) and mn edges (one for each geometric square in X). Finally, the amalgamated group $F_{1-2m+2mn}$ is the fundamental group of a graph having $2m$ vertices (one for each edge in E_h) and $2mn$ edges (one for each geometric square in the vertically subdivided complex X'). The two injections in the amalgamated free product are induced by immersions in X'. Analogous statements hold for the second splitting of Γ.

Proposition 3. *Let Γ be a $(2m, 2n)$-group. We denote by $F_n^{(2)}$ the subgroup of $F_n = \langle b_1, \ldots, b_n\rangle$ of index 2 consisting of elements with even length. Analogously, we define $F_m^{(2)} \lhd F_m = \langle a_1, \ldots, a_m\rangle$. If $\rho_v(F_n^{(2)}) < S_{2m}$ is transitive (which is satisfied if for example P_h is a primitive permutation group), then there is an amalgam decomposition of Γ_0 (the so-called* vertical decomposition *of Γ_0)*

$$\Gamma_0 \cong F_{2n-1} *_{F_{1-4m+4mn}} F_{2n-1}.$$

If $\rho_h(F_m^{(2)}) < S_{2n}$ is transitive (which is satisfied if for example P_v is primitive), then we get a horizontal decomposition

$$\Gamma_0 \cong F_{2m-1} *_{F_{1-4n+4mn}} F_{2m-1}.$$

In particular, if $m = n$ and $\rho_v(F_n^{(2)})$, $\rho_h(F_m^{(2)})$ both are transitive, then we have two decompositions of Γ_0 as

$$F_{2n-1} *_{F_{(2n-1)^2}} F_{2n-1}.$$

Proof. Again, this can be directly deduced from the more general result [69, Theorem I.1.18]. $\square$

We call a $(2m, 2n)$-group Γ *horizontally directed*, if a_i is not in the same orbit as a_i^{-1} in the natural action of P_h on E_h for all $i \in \{1, \ldots, m\}$. The term *vertically directed* is defined analogously. These definitions are equivalent to those given in [69, Definition I.1.10]. We formulate another interesting special case of [69, Theorem I.1.18] concerning HNN-extensions:

Proposition 4. *Let Γ be a $(2m, 2n)$-group.*

(1*a*) *If Γ is horizontally directed and P_h has exactly two orbits in its natural action on E_h, then Γ is a HNN-extension of the free group $F_n = \langle b_1, \ldots, b_n \rangle$ associating subgroups F_{1-m+mn} of index m.*

(1*b*) *If Γ is vertically directed and P_v has exactly two orbits in its natural action on E_v, then Γ is a HNN-extension of the free group $F_m = \langle a_1, \ldots, a_m \rangle$ associating subgroups F_{1-n+mn} of index n.*

Remark. Horizontally directed $(2m, 2n)$-groups Γ satisfy $|\Gamma^{ab}| = \infty$, in particular they have a proper infinite quotient. To see this, let $\mathcal{O}_1$ be the orbit of a_1 under the natural action of P_h on E_h. Define $\Gamma \to \mathbb{Z}$ by sending all elements in $\mathcal{O}_1$ to the generator 1 of $\mathbb{Z}$. If both a_i and a_i^{-1} are not in $\mathcal{O}_1$, then we map a_i, as well as all b_j, $j = 1, \ldots, n$, to the trivial element 0 in $\mathbb{Z}$. The same statement holds for vertically directed $(2m, 2n)$-groups.

Definition. Let X be a $(2m, 2n)$-complex and Y a $(2\tilde{m}, 2\tilde{n})$-complex, where $\tilde{m} \geq m$ and $\tilde{n} \geq n$. We say that X is *embedded* in Y, if the $\tilde{m}\tilde{n}$ geometric squares of Y contain all mn geometric squares of X.

Proposition 5. *Let the $(2m, 2n)$-complex X be embedded in the $(2\tilde{m}, 2\tilde{n})$-complex Y, where $\tilde{m} \geq m$ and $\tilde{n} \geq n$. Then*

(1) $\pi_1 X < \pi_1 Y$.

(2) $|P_h^{(k)}(X)|$ *divides* $|P_h^{(k)}(Y)|$ *and* $|P_v^{(k)}(X)|$ *divides* $|P_v^{(k)}(Y)|$ *for each* $k \in \mathbb{N}$.

(3) *If X is irreducible then Y is irreducible.*

Proof. (1) See [8, Proposition II.4.14(1)].

(2) To take into account the two involved complexes X and Y, we write here $P_h^{(k)}(X)$, $P_h^{(k)}(Y)$, $P_v^{(k)}(X)$, $P_v^{(k)}(Y)$, $\rho_{v,X}$, $\rho_{v,Y}$ instead of $P_h^{(k)}$, $P_v^{(k)}$, ρ_v. We prove now that $|P_h(X)|$ divides $|P_h(Y)|$. The other statements are proved similarly. Let G be the subgroup of $S_{2\tilde{m}}$

$$G := \langle \rho_{v,Y}(b_1), \ldots, \rho_{v,Y}(b_n) \rangle_{S_{2\tilde{m}}}$$

and Δ the subset of $\{1, \ldots, 2\tilde{m}\}$ with $2m$ elements

$$\Delta := \{1, \ldots, m, 2\tilde{m} - m + 1, \ldots, 2\tilde{m}\}.$$

Because of the embedding assumption and the link conditions in X and Y, Δ is G-invariant and the restriction of G to Δ is permutation isomorphic to

$$P_h(X) = \langle \rho_{v,X}(b_1), \ldots, \rho_{v,X}(b_n) \rangle_{S_{2m}}$$

via the inclusion

$$\begin{aligned} \{1, \ldots, 2m\} &\to \{1, \ldots, 2\tilde{m}\} \\ i &\mapsto i \\ 2m + 1 - i &\mapsto 2\tilde{m} + 1 - i, \end{aligned}$$

$i = 1, \ldots, m$, hence $|G| = |P_h(X)| \cdot \ell$, where ℓ is the order of the pointwise stabilizer of Δ in G (cf. [24, p. 17]). The claim follows now, since G is obviously a subgroup of

$$\langle \rho_{v,Y}(b_1), \ldots, \rho_{v,Y}(b_n), \ldots, \rho_{v,Y}(b_{\tilde{n}}) \rangle_{S_{2\tilde{m}}} = P_h(Y).$$

(3) The set $\{|P_h^{(k)}(X)|\}_{k\in\mathbb{N}}$ is unbounded since X is irreducible by assumption, hence by part (2) also $\{|P_h^{(k)}(Y)|\}_{k\in\mathbb{N}}$ is unbounded, i.e. Y is irreducible, too.

□

Because of the link condition in X, every element $\gamma \in \Gamma$ can be brought in a unique normal form, where the "a's are followed by b's" and in a unique normal form, where the "b's are followed by a's". The following proposition is due to Bridson and Wise (see [9]).

Proposition 6. *(see [9, Normal Form Lemma 4.3]) Let Γ be a $(2m, 2n)$-group and $\gamma \in \Gamma$ any element. Then γ can be written as*

$$\gamma = \sigma_a\sigma_b = \sigma'_b\sigma'_a$$

*where σ_a, σ'_a are freely reduced words in the subgroup $\langle a_1, \ldots, a_m\rangle$ and σ_b, σ'_b are freely reduced words in $\langle b_1, \ldots, b_n\rangle$. The words $\sigma_a, \sigma'_a, \sigma_b, \sigma'_b$ are uniquely determined by γ. Moreover, $|\sigma_a| = |\sigma'_a|$ and $|\sigma_b| = |\sigma'_b|$, where $|\cdot|$ is the word length with respect to the standard generators, in particular $|1| = 0$. We call $\sigma_a\sigma_b$ the ab-*normal form *and $\sigma'_b\sigma'_a$ the ba-*normal form *of γ. The* length *of γ is by definition $|\gamma| := |\sigma_a| + |\sigma_b| = |\sigma'_b| + |\sigma'_a|$.*

Proof. See [9]. For an implementation in GAP ([28]) to compute the two normal forms of a given element in Γ, see Appendix D.6. □

Corollary 7. *Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m, n)\rangle$ be a $(2m, 2n)$-group. Then*

(1) *$\langle a_1, \ldots, a_m\rangle$ and $\langle b_1, \ldots, b_n\rangle$ are free subgroups of Γ.*

(2) *Γ is virtually abelian or contains a non-abelian free subgroup.*

(3) *The center $Z\Gamma$ is trivial if $\min\{m, n\} \geq 2$.*

(4) *Γ is residually finite if and only if $\mathrm{Aut}(\Gamma)$ is residually finite.*

Proof. (1) This follows from the uniqueness of the normal forms in Proposition 6.

(2) If $m \geq 2$ or $n \geq 2$ then Γ contains a non-abelian free subgroup by (1). If $m = n = 1$, then either

$$\Gamma \cong \langle a_1, b_1 \mid a_1b_1 = b_1a_1\rangle \cong \mathbb{Z}^2,$$

or

$$\Gamma \cong \langle a_1, b_1 \mid a_1b_1a_1 = b_1\rangle,$$

which has the abelian subgroup $\langle a_1, b_1^2\rangle \cong \mathbb{Z}^2$ of index 2.

(3) Assume that there is an element $\gamma \in Z\Gamma \setminus \{1\}$ and let

$$\gamma = a^{(1)} \ldots a^{(k)}b^{(1)} \ldots b^{(l)}$$

be its ab-normal form, where we can assume without loss of generality that $k \geq 1$ and $l \geq 0$. Take

$$a \in \{a_1, \ldots a_m, a_m^{-1}, \ldots, a_1^{-1}\} \setminus \{a^{(1)}, {a^{(1)}}^{-1}\} \neq \emptyset.$$

Then, we have

$$aa^{(1)} \ldots a^{(k)}b^{(1)} \ldots b^{(l)} = a^{(1)} \ldots a^{(k)}b^{(1)} \ldots b^{(l)}a.$$

The left hand side is already in ab-normal form, hence by uniqueness of the ab-normal form, we can conclude $a = a^{(1)}$, but this is a contradiction to the choice of a, i.e. $Z\Gamma = 1$.

(4) By a result of Gilbert Baumslag ([4] or see [47, Theorem IV.4.8]) the group $\mathrm{Aut}(\Gamma)$ is residually finite, if Γ is a finitely generated residually finite group. For the other direction, first note that if $m = 1$, then

$$P_h^{(k)} < S_{2m\cdot(2m-1)^{k-1}} = S_2,$$

hence $|P_h^{(k)}| \leq 2$, $k \in \mathbb{N}$, and Γ is reducible. The same holds if $n = 1$. In particular, Γ is residually finite, if $m = 1$ or $n = 1$. Assume now that Γ is non-residually finite, then $\min\{m, n\} \geq 2$, and by (2) we have $Z\Gamma = 1$, hence $\Gamma \cong \mathrm{Inn}(\Gamma) < \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ is non-residually finite.

□

Remark. $\mathbb{Z} \times F_n$ is a $(2, 2n)$-group with a non-trivial center.

Proposition 8. *Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m, n)\rangle$ be a $(2m, 2n)$-group.*

(1a) *Assume that there is an element $a_i \in \{a_1, \ldots, a_m\}$ such that $\rho_h(a_i)(b) \neq b$ for all $b \in E_v$ (i.e. $R(m, n)$ has no relator equivalent to a relator of the form $a_i bab^{-1}$ for all $a \in E_h$ and $b \in E_v$). Then $Z_\Gamma(a_i) = \langle a_i\rangle$.*

(1b) *Assume that there is an element $b_j \in \{b_1, \ldots, b_n\}$ such that $\rho_v(b_j)(a) \neq a$ for all $a \in E_h$ (i.e. $R(m, n)$ has no relator equivalent to a relator of the form $a^{-1}b_j ab$ for all $a \in E_h$ and $b \in E_v$). Then $Z_\Gamma(b_j) = \langle b_j\rangle$.*

(2a) *Assume that $Z_\Gamma(a_i) = \langle a_i\rangle$ for some $a_i \in \{a_1, \ldots, a_m\}$. Then $N_\Gamma(\langle a_i\rangle) = \langle a_i\rangle$.*

(2b) *Assume that $Z_\Gamma(b_j) = \langle b_j\rangle$ for some $b_j \in \{b_1, \ldots, b_n\}$. Then $N_\Gamma(\langle b_j\rangle) = \langle b_j\rangle$.*

Proof. We prove (1b) and (2b), the proofs of (1a) and (2a) are similar.

(1b) Obviously, $\langle b_j\rangle < Z_\Gamma(b_j)$. We have to show $Z_\Gamma(b_j) < \langle b_j\rangle$. Let $\gamma = a^{(1)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} \in Z_\Gamma(b_j)$ be in ab-normal form, $k, l \geq 0$. Then

$$a^{(1)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} b_j = b_j a^{(1)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)}.$$

Assume first that $k \geq 1$. The ab-normal form of γb_j starts with $a^{(1)} \ldots a^{(k)}$. Bringing also $b_j a^{(1)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)}$ to this normal form, we must have in a first step $b_j a^{(1)} = a^{(1)} b$ for some $b \in E_v$, i.e. $\rho_v(b_j)(a^{(1)}) = a^{(1)}$, which is impossible by assumption, hence $k = 0$. This means $\gamma = b^{(1)} \ldots b^{(l)}$ and

$$b^{(1)} \ldots b^{(l)} b_j = b_j b^{(1)} \ldots b^{(l)}.$$

By uniqueness of the ab-normal form of

$$b_j = {b^{(l)}}^{-1} \ldots {b^{(1)}}^{-1} b_j b^{(1)} \ldots b^{(l)}$$

we have $l = 0$ or $b^{(1)}, \ldots, b^{(l)} \in \{b_j, b_j^{-1}\}$ and hence $\gamma = b^{(1)} \ldots b^{(l)} \in \langle b_j\rangle$.

(2b) Obviously, $\langle b_j\rangle < N_\Gamma(\langle b_j\rangle)$. We have to show $N_\Gamma(\langle b_j\rangle) < \langle b_j\rangle$. Let $\gamma \in N_\Gamma(\langle b_j\rangle)$, then in particular $\gamma^{-1} b_j \gamma \in \langle b_j\rangle$, i.e. b_j is conjugate to a power of itself, hence by a result of Bridson and Haefliger (see Proposition 17) we conclude $\gamma^{-1} b_j \gamma \in \{b_j, b_j^{-1}\}$. If $\gamma^{-1} b_j \gamma = b_j$, then $\gamma \in Z_\Gamma(b_j) = \langle b_j\rangle$ and we are done. So from now on let us suppose that $\gamma^{-1} b_j \gamma = b_j^{-1}$ (we will see in the proof that this case is in fact not possible under the assumption $Z_\Gamma(b_j) = \langle b_j\rangle$), then

$$\gamma^{-2} b_j \gamma^2 = \gamma^{-1}(\gamma^{-1} b_j \gamma)\gamma = \gamma^{-1} b_j^{-1} \gamma = (\gamma^{-1} b_j \gamma)^{-1} = (b_j^{-1})^{-1} = b_j,$$

i.e. $\gamma^2 \in Z_\Gamma(b_j) = \langle b_j \rangle$ (which however does *not* directly imply $\gamma \in \langle b_j \rangle$ in general). Let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)},$$

$k, l \geq 0$, be the ab-normal form of γ. We assume first that $k \geq 1$, in particular $\gamma \neq 1$. Then

$$\gamma^2 = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} = b_j^s \tag{1}$$

for some $s \in \mathbb{Z} \setminus \{0\}$ (we know that $s \neq 0$, since Γ is torsion-free and $\gamma \neq 1$). Note that it follows $l \geq 1$, otherwise we would have the contradiction $(a^{(1)} \dots a^{(k)})^2 = b_j^s$. The expression $b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)}$ is in ba-normal form, let $\tilde{a}^{(k)} \dots \tilde{a}^{(1)} \tilde{b}^{(1)} \dots \tilde{b}^{(l)}$ be its ab-normal form, i.e.

$$b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)} = \tilde{a}^{(k)} \dots \tilde{a}^{(1)} \tilde{b}^{(1)} \dots \tilde{b}^{(l)}. \tag{2}$$

Then, putting (2) into (1) gives

$$\gamma^2 = a^{(1)} \dots a^{(k)} \tilde{a}^{(k)} \dots \tilde{a}^{(1)} \tilde{b}^{(1)} \dots \tilde{b}^{(l)} b^{(1)} \dots b^{(l)} = b_j^s. \tag{3}$$

The right hand side b_j^s of equation (3) is in ab-normal form, hence the a's on the left hand side have to cancel (i.e. $\tilde{a}^{(k)} = {a^{(k)}}^{-1}, \dots, \tilde{a}^{(1)} = {a^{(1)}}^{-1}$, because $a^{(1)} \dots a^{(k)}$ and $\tilde{a}^{(k)} \dots \tilde{a}^{(1)}$ are freely reduced words in $\langle a_1, \dots, a_m \rangle$), so we have

$$b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)} = {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} \tilde{b}^{(1)} \dots \tilde{b}^{(l)} \tag{4}$$

from equation (2) and

$$\gamma^2 = \tilde{b}^{(1)} \dots \tilde{b}^{(l)} b^{(1)} \dots b^{(l)} = b_j^s \tag{5}$$

from equation (3). Moreover, since $b^{(1)} \dots b^{(l)}$ and $\tilde{b}^{(1)} \dots \tilde{b}^{(l)}$ are freely reduced words in $\langle b_1, \dots, b_n \rangle$, we conclude from equation (5) that s is even,

$$b^{(1)} \dots b^{(l)} = b^{(1)} \dots b^{(r)} b_j^t \tag{6}$$

and

$$\tilde{b}^{(1)} \dots \tilde{b}^{(l)} = b_j^t {b^{(r)}}^{-1} \dots {b^{(1)}}^{-1}, \tag{7}$$

where $t = s/2$ and $0 \leq r < l$ is the number of cancellations in $\tilde{b}^{(1)} \dots \tilde{b}^{(l)} b^{(1)} \dots b^{(l)}$, i.e. $\tilde{b}^{(l)} b^{(1)} = 1, \dots, \tilde{b}^{(l-r+1)} b^{(r)} = 1$. Note that $|t| = l - r \geq 1$, in particular also the right hand sides of (6) and (7) are in normal form. First, we assume $r \geq 1$. Putting (6) and (7) into (4), we get

$$b^{(1)} \dots b^{(r)} b_j^t a^{(1)} \dots a^{(k)} = {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j^t {b^{(r)}}^{-1} \dots {b^{(1)}}^{-1}. \tag{8}$$

Since both sides of equation (8) are in normal form, we have (looking at the right ends)

$$b_j^{\pm 1} a^{(1)} \dots a^{(k)} = w_k(a) {b^{(1)}}^{-1} \tag{9}$$

and (looking at the left ends)

$${a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j^{\pm 1} = b^{(1)} \tilde{w}_k(a), \tag{10}$$

where $w_k(a)$ and $\tilde{w}_k(a)$ are freely reduced words of length k in $\langle a_1, \dots, a_m \rangle$, and the sign of b_j in (9) and (10) is according to the sign of t, i.e. we have b_j, if t is positive, and b_j^{-1}, if t is negative. Now, equation (10) gives

$$a^{(1)} \dots a^{(k)} = b_j^{\pm 1} \tilde{w}_k^{-1}(a) {b^{(1)}}^{-1}. \tag{11}$$

Putting (11) into (9) gives

$$b_j^{\pm 2}\tilde{w}_k^{-1}(a){b^{(1)}}^{-1} = w_k(a){b^{(1)}}^{-1}, \tag{12}$$

i.e. the contradiction $b_j^{\pm 2} = w_k(a)\tilde{w}_k(a) \in \langle a_1, \dots, a_m \rangle$. Thus, we have to study the remaining case $r = 0$, i.e. $|t| = l = |s|/2$ and

$$\gamma = a^{(1)} \dots a^{(k)} b_j^t.$$

Then equation (4) or (8) is

$$b_j^t a^{(1)} \dots a^{(k)} = {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j^t, \tag{13}$$

which is equivalent to

$${a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j = b_j^t a^{(1)} \dots a^{(k)} b_j^{1-t}. \tag{14}$$

The equation $\gamma^{-1} b_j \gamma = b_j^{-1}$ is equivalent to

$$b_j^{-t} {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j a^{(1)} \dots a^{(k)} b_j^t = b_j^{-1}. \tag{15}$$

Putting (14) into (15) gives

$$b_j^{-t} b_j^t a^{(1)} \dots a^{(k)} b_j^{1-t} a^{(1)} \dots a^{(k)} b_j^t = b_j^{-1} \tag{16}$$

or equivalently

$$a^{(1)} \dots a^{(k)} b_j^{1-t} = b_j^{-1-t} {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1}, \tag{17}$$

which is a contradiction, since both sides of the equation are in normal form, but $t = s/2 \neq 0$ and hence $|b_j^{1-t}| = |1 - t| \neq |-1-t| = |b_j^{-1-t}|$. This means that the case $k \geq 1$ is impossible. It remains to consider the case $k = 0$, i.e. $\gamma = b^{(1)} \dots b^{(l)}$ for some $l \geq 0$. But then, $\gamma^{-1} b_j \gamma = b_j^{-1}$ gives a non-trivial relation in the free group $\langle b_1, \dots, b_n \rangle$.

□

Remark. The assumptions of Proposition 8(1a),(1b) are not necessary as shown in Theorem 1(9).

We state now an adapted version of the crucial "normal subgroup theorem" due to Burger and Mozes ([14], [15], [16]).

Proposition 9. *(see [16, Chapter 4 and 5]) Let Γ be an irreducible $(2m, 2n)$-group such that P_h, P_v are 2-transitive, and $\mathrm{Stab}_{P_h}(\{1\})$, $\mathrm{Stab}_{P_v}(\{1\})$ are non-abelian simple groups. Then any non-trivial normal subgroup of Γ has finite index in Γ.*

Proof. This is a combination of [16, Corollary 5.1], [16, Proposition 5.2] and [16, Corollary 5.3]. □

Remark. We will apply Proposition 9 to irreducible $(2m, 2n)$-groups such that

$$(P_h, P_v) \in \{(A_{2m}, A_{2n}), (A_{2m}, M_{12}), (A_{2m}, \mathrm{ASL}_3(2)), (M_{12}, A_{2n}), (\mathrm{ASL}_3(2), A_{2n})\},$$

where $m \geq 3$, $n \geq 3$, $M_{12} < S_{12}$, $\mathrm{ASL}_3(2) < S_8$ (cf. [15, Chapter 3.3]).

Some general notations for groups: The trivial group is usually denoted by 1. For a group G and a subset $S \subseteq G$, let $\langle S \rangle$ be the subgroup of G generated by S and let $\langle\!\langle S \rangle\!\rangle_G$ be the normal closure of S in G, i.e. the smallest normal subgroup of G containing S. We denote by G^{ab} the abelianization $G/[G, G]$, by $|G|$ the order of G, by $Z(G)$ or ZG the center of G, and by $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ the commutator of two elements $g_1, g_2 \in G$. Both signs $<$ and $\lhd$ do not exclude equality. Let $g \in G$ and $M < G$ a subgroup. Then, we denote by $Z_G(g)$ the centralizer $\{h \in G : hg = gh\}$ and by $N_G(M)$ the normalizer $\{h \in G : hMh^{-1} = M\}$. A subgroup $M < G$ is called proper, if $M \neq G$. For a normal subgroup $N \lhd G$, a quotient G/N is called proper if $N \neq 1$. Let $d(G)$ be the minimal number of generators of G ($d(G) = \infty$, if G is not finitely generated). Let G^k be the direct product $G \times \ldots \times G$ of k copies of G and G^{*k} the free product $G * \ldots * G$ of k copies of G

Let G be a *permutation group*, i.e. $G < \mathrm{Sym}(\Omega)$ for some non-empty set Ω. For $k \in \mathbb{N}$, the group G is said to be *k-transitive* if for every pair $(\omega_1, \ldots, \omega_k)$, $(\xi_1, \ldots, \xi_k)$ of k-tuples of distinct points in Ω, there exists an element $g \in G$ such that $g(\omega_1) = \xi_1, \ldots, g(\omega_k) = \xi_k$. Let $G < \mathrm{Sym}(\Omega)$ be a transitive (i.e. 1-transitive, according to the given definition) permutation group. A non-empty subset $\Delta \subseteq \Omega$ is called a *block* for G, if for each $g \in G$ either $g(\Delta) = \Delta$ or $g(\Delta) \cap \Delta = \emptyset$. We say that G is *primitive* if it has no non-trivial blocks on Ω, i.e. no blocks except Ω itself and the one-element subsets $\{\omega\}$ of Ω. Two permutation groups $G < \mathrm{Sym}(\Omega)$ and $H < \mathrm{Sym}(\Omega')$ are called *permutation isomorphic* if there exists a bijection $f : \Omega \to \Omega'$ and a group isomorphism $\psi : G \to H$ such that the following diagram commutes for each $g \in G$

$$\begin{tikzcd}
\Omega \arrow[r, "g"] \arrow[d, "f"'] & \Omega \arrow[d, "f"] \\
\Omega' \arrow[r, "\psi(g)"] & \Omega'
\end{tikzcd}$$

Further notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Sometimes, by abuse of notations, we write x instead of $\{x\}$.

2 Irreducible (A_6, P_v)-groups, P_v primitive

In this section, we want to construct irreducible (A_6, P_v)-groups, where $P_v < S_{2n}$ is a primitive permutation group. For practical reasons, we restrict to $3 \leq n \leq 7$. This range covers 30 different groups P_v (up to isomorphism). See Table 1 for a short description of them and see [12] for a list of all finite primitive permutation groups up to degree 50. A comprehensive introduction to permutation groups, including the definitions of the groups in Table 1, is given in [24]. We examine in more details four examples: (A_6, A_6)-group in Example 1, $(A_6, S_5 < S_{10})$-group in Example 3, (A_6, M_{12})-group in Example 6 and $(A_6, \mathrm{ASL}_3(2))$-group in Example 7. Two examples, $(A_6, A_5 < S_{10})$ and $(A_6, M_{11} < S_{12})$, could not be found with our methods up to now. A construction of the remaining 24 examples is presented in Appendix A.1. We would like to justify our choice $P_h = A_6$. First, Proposition 1(1a) provides us with a necessary and sufficient condition for irreducibility. Applying this, our examples in Section 2 and Appendix A.1 are irreducible, since in all cases $|P_h^{(2)}| = 360 \cdot 60^6$. The second reason is that we know H_1: it is the "universal group" $U(A_6) < \mathrm{Aut}(\mathcal{T}_6)$. This enables us to conclude in some cases (under strong assumptions on P_v, see Proposition 9), that Γ has no non-trivial normal subgroups of infinite index. Being also of independent interest, this result is a first step in the construction of finitely presented torsion-free (virtually) simple groups. Each example will be given in terms of the set of relators (geometric squares) $R(m,n)$. This uniquely determines the group

$$\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n) \rangle,$$

as well as the complex X, which is the standard 2-dimensional cell complex associated with Γ. In particular, X is a $(2m, 2n)$-complex such that $\pi_1(X, x) = \Gamma$, where x is as usual the single vertex of X. In all our examples, we use the same names for the appearing groups and spaces: Γ, X, Γ_0, P_h, P_v, H_1, H_2, ..., as introduced in Section 1. This should not lead to confusion, since they always refer to the last defined example.

2.1 (A_6, A_6)-group

Example 1.

$$R(3,3) := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, & a_1 b_3 a_2 b_2^{-1}, \\ a_1 b_3^{-1} a_3^{-1} b_2, & a_2 b_1 a_3^{-1} b_2^{-1}, & a_2 b_2 a_3^{-1} b_3^{-1}, \\ a_2 b_3 a_3^{-1} b_1, & a_2 b_3^{-1} a_3 b_2, & a_2 b_1^{-1} a_3^{-1} b_1^{-1} \end{array} \right\}.$$

Theorem 1. (1) *$P_h = A_6$, $P_v = A_6$.*

(2) *Any non-trivial normal subgroup of Γ has finite index.*

(3) *Γ can be decomposed in two different ways as an amalgamated free product of finitely generated free groups $\Gamma \cong F_3 *_{F_{13}} F_7$. The same holds for its subgroup $\Gamma_0 \cong F_5 *_{F_{25}} F_5$.*

(4) *$\Gamma \cong \mathrm{pr}_i(\Gamma) \lneqq H_i$, $i = 1, 2$.*

(5) *The second bounded cohomology of Γ with $\mathbb{R}$-coefficients vanishes, i.e. $H_b^2(\Gamma; \mathbb{R}) = 0$ (cf. Theorem 3(3)).*

(6) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(7) *(cf. [16, Theorem 6.3] where $m \geq 15$, $n \geq 19$) For every $m \geq 7$ and $n \geq 7$, there exists a torsion-free cocompact lattice $\Lambda < U(A_{2m}) \times U(A_{2n})$ with dense projections. Any non-trivial normal subgroup $N \lhd \Lambda$ is of finite index in Λ.*

G	$2n$	$\mathrm{t}(G)$	$\lvert G\rvert$	$G < A_{2n}$
$\mathrm{PSL}_2(5)$	6	2	60	Y
$\mathrm{PGL}_2(5)$	6	3	120	N
A_6	6	4	360	Y
S_6	6	6	720	N
$\mathrm{AGL}_1(8)$	8	2	56	Y
$\mathrm{A\Gamma L}_1(8)$	8	2	168	Y
$\mathrm{PSL}_2(7)$	8	2	168	Y
$\mathrm{PGL}_2(7)$	8	3	336	N
$\mathrm{ASL}_3(2)$	8	3	1344	Y
A_8	8	6	20160	Y
S_8	8	8	40320	N
A_5	10	1	60	Y
S_5	10	1	120	N
$\mathrm{PSL}_2(9)$	10	2	360	Y
S_6	10	2	720	N
$\mathrm{PGL}_2(9)$	10	3	720	N
M_{10}	10	3	720	Y
$\mathrm{P\Gamma L}_2(9)$	10	3	1440	N
A_{10}	10	8	$10!/2$	Y
S_{10}	10	10	$10!$	N
$\mathrm{PSL}_2(11)$	12	2	660	Y
$\mathrm{PGL}_2(11)$	12	3	1320	N
M_{11}	12	3	7920	Y
M_{12}	12	5	95040	Y
A_{12}	12	10	$12!/2$	Y
S_{12}	12	12	$12!$	N
$\mathrm{PSL}_2(13)$	14	2	1092	Y
$\mathrm{PGL}_2(13)$	14	3	2184	N
A_{14}	14	12	$14!/2$	Y
S_{14}	14	14	$14!$	N

Table 1: Primitive subgroups of S_{2n}, $3 \leq n \leq 7$. $\mathrm{t}(G)$: transitivity of G.

(8) $\mathrm{Aut}(X) \cong \mathbb{Z}_2$.

(9) $Z_\Gamma(a_i) = N_\Gamma(\langle a_i\rangle) = \langle a_i\rangle$, *if* $a_i \in \{a_2, a_3\}$. $Z_\Gamma(b_j) = N_\Gamma(\langle b_j\rangle) = \langle b_j\rangle$, *if* $b_j \in \{b_2, b_3\}$.

(10) Γ *is not linear over any field.*

Proof. (1) We only list the generators of P_h and P_v. It can easily be checked for example with

GAP ([28]), that these permutations indeed generate A_6.

$$\begin{aligned}
\rho_v(b_1) &= (2,3)(4,5),\\
\rho_v(b_2) &= (1,5,4,2,3),\\
\rho_v(b_3) &= (2,3,5,4,6), \text{ generating } P_h = A_6.
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (2,3)(4,5),\\
\rho_h(a_2) &= (1,6,3,2)(4,5),\\
\rho_h(a_3) &= (1,4,5,6)(2,3), \text{ generating } P_v = A_6.
\end{aligned}$$

(2) We apply Proposition 9 or [16, Corollary 5.3], using the fact that P_h and P_v are 2-transitive, $\mathrm{Stab}_{P_h}(\{1\}) = \langle (2,3)(4,5), (2,3,5,4,6)\rangle \cong A_5$, $\mathrm{Stab}_{P_v}(\{1\}) = \langle (2,3)(4,5), (2,4,5), (4,5,6)\rangle \cong A_5$ are non-abelian simple groups and that Γ is irreducible.

(3) See [69, Theorem I.1.18]. Explicit (i.e. describing the injections) decompositions of Γ and Γ_0 are given in Appendix A.2.

(4) By [15, Proposition 3.1.2], $\mathrm{QZ}(H_i) = 1$ for $i = 1, 2$, hence pr_{3-i} is injective showing $\Gamma \cong \mathrm{pr}_i(\Gamma)$. H_i is by [15, Proposition 3.3.1] isomorphic to the "universal group" $U(A_6)$, which is not torsion-free, thus $\mathrm{pr}_i(\Gamma) \neq H_i$. For a definition and some properties of the universal group, see [15, Chapter 3.2] or [16, Chapter 5].

(5) We have noticed in the proof of (4) that $H_i \cong U(A_6)$, $i = 1, 2$. Hence, by [15, Chapter 3], H_1 and H_2 act transitively on the boundary at infinity $\partial_\infty \mathcal{T}_6$ of their corresponding trees $\mathcal{T}_{2m} = \mathcal{T}_6$ and $\mathcal{T}_{2n} = \mathcal{T}_6$ respectively. Now, statement (5) follows from [13, Corollary 26]. As pointed out there, this result has some applications to Γ-actions on the circle S^1 (see [13, Corollary 22]).

(6) These are easy computations using GAP ([28]). To see by hand that Γ_0 is perfect, one first computes a presentation of Γ_0 by the Reidemeister-Schreier method and then adds commutators to the relators to simplify the presentation.

(7) We follow the proof of [16, Theorem 6.3], but replace the $(\mathrm{PSL}_2(13), \mathrm{PSL}_2(17))$-complex ${}^{(0)}X = \mathcal{A}_{13,17}$ used there (see also Example 40) by our (A_6, A_6)-complex X. An illustration of this construction is given in Appendix A.3 for the smallest values $m = 7$, $n = 7$.

(8) An *automorphism* of X is a a graph automorphism of the 1-skeleton $X^{(1)}$ which induces a permutation on the set of geometric squares of X. Checking all $2^6 6! = 46080$ candidates (using the program of Appendix D.7), we have found exactly one non-trivial automorphism (fixing seven of nine geometric squares) given by

$$\begin{aligned}
a_1 &\mapsto a_1^{-1}\\
a_2 &\mapsto a_2^{-1}\\
a_3 &\mapsto a_3^{-1}\\
b_1 &\mapsto b_1^{-1}\\
b_2 &\mapsto b_3\\
b_3 &\mapsto b_2.
\end{aligned}$$

The two non-trivially permuted geometric squares of X are $a_2 b_1 a_3^{-1} b_2^{-1}$ and $a_2 b_3 a_3^{-1} b_1$. Note that this automorphism gives a non-trivial element in the group of outer automorphisms $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$, since it has order 2 and $\mathrm{Inn}(\Gamma) \cong \Gamma$ is torsion-free (the isomorphism $\mathrm{Inn}(\Gamma) \cong \Gamma$ holds because $Z\Gamma = 1$ by Corollary 7(3)).

(9) $Z_\Gamma(a_2) = N_\Gamma(\langle a_2\rangle) = \langle a_2\rangle$, $Z_\Gamma(a_3) = N_\Gamma(\langle a_3\rangle) = \langle a_3\rangle$, $N_\Gamma(\langle b_2\rangle) = \langle b_2\rangle$ and $N_\Gamma(\langle b_3\rangle) = \langle b_3\rangle$ follow from Proposition 8. We prove $Z_\Gamma(b_3) = \langle b_3\rangle$. Similarly, one can prove $Z_\Gamma(b_2) = \langle b_2\rangle$. Let $\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} \in Z_\Gamma(b_3)$ be in ab-normal form, $k, l \geq 0$. Then

$$a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} b_3 = b_3 a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)}.$$

Assume first that $k = 1$, i.e.

$$a^{(1)} b^{(1)} \dots b^{(l)} b_3 = b_3 a^{(1)} b^{(1)} \dots b^{(l)}.$$

The ab-normal form of $a^{(1)} b^{(1)} \dots b^{(l)} b_3$ starts with $a^{(1)}$. Bringing also $b_3 a^{(1)} b^{(1)} \dots b^{(l)}$ to this normal form, we must have in a first step $b_3 a^{(1)} = a^{(1)} b$ for some $b \in E_v$. Checking all elements in $R(3,3)$, the only possibility is $a^{(1)} = a_1$, $b = b_2$, hence

$$a_1 b^{(1)} \dots b^{(l)} b_3 = a_1 b_2 b^{(1)} \dots b^{(l)}$$

or

$$b^{(1)} \dots b^{(l)} b_3 = b_2 b^{(1)} \dots b^{(l)}.$$

But this gives a non-trivial relation in the free group $\langle b_1, b_2, b_3\rangle$.

Assume now that $k \geq 2$. As in the case $k = 1$, we conclude $a^{(1)} = a_1$ and $b_3 a^{(1)} = a^{(1)} b_2$, i.e.

$$a_1 a^{(2)} \dots a^{(k)} b^{(1)} \dots b^{(l)} b_3 = a_1 b_2 a^{(2)} \dots a^{(k)} b^{(1)} \dots b^{(l)}$$

hence

$$a^{(2)} \dots a^{(k)} b^{(1)} \dots b^{(l)} b_3 = b_2 a^{(2)} \dots a^{(k)} b^{(1)} \dots b^{(l)}$$

The ab-normal form of the left hand side of this equation starts with $a^{(2)}$. Bringing the right hand side to this normal form, we must have $b_2 a^{(2)} = a^{(2)} b$ for some $b \in E_v$. Here, the only possibility is $a^{(2)} = a_1^{-1}$, $b = b_3$, but this contradicts the fact that $a^{(1)} a^{(2)} \dots a^{(k)} = a_1 a_1^{-1} \dots a^{(k)}$ is freely reduced.

It follows that $k = 0$, and we conclude $\gamma \in \langle b_3\rangle$ exactly as in the proof of Proposition 8(1b).

(10) It follows from Proposition 38 in Section 4.9.

$\square$

Conjecture 1. *The finitely presented torsion-free group Γ_0 is simple.*

A possible proof of Conjecture 1 could use the following easy lemmas:

Lemma 10. *Let G be a group and $H < G$ a subgroup of finite index. Then there is a group $N < H$ such that $N \lhd G$ and $[G : N] \leq [G : H]! < \infty$, in particular*

$$\bigcap_{M \overset{f.i.}{<} G} M = \bigcap_{L \overset{f.i.}{\lhd} G} L.$$

Proof. (Probably due to Marshall Hall Jr. ([30])) Let k be the index $[G : H]$ and write G as a disjoint union of right cosets

$$G = \bigsqcup_{i=1}^{k} H g_i.$$

Right multiplication $Hg_i \mapsto Hg_i g$ induces a homomorphism $\phi : G \to S_k$ such that $N := \ker\phi < H$ and $[G : N] \leq |S_k| = [G : H]! < \infty$. Note that

$$N = \bigcap_{g \in G} gHg^{-1}.$$

$\square$

Lemma 11. *Let G be a group and $H \lhd G$ a normal subgroup of finite index. Assume that there is an element $h \in H$ such that $\langle\langle h^k \rangle\rangle_G > H$ for each $k \in \mathbb{N}$. Then every proper normal subgroup of H has infinite index.*

Proof. Let $N \lhd H$ be a normal subgroup of finite index. By Lemma 10, there is a group $M < N$ such that $M \lhd G$ and $[G : M] < \infty$. Looking at cosets of the form $h^k M$, $k \in \mathbb{N}$, we see that at least two of them are equal, in particular $h^i \in M$ for an $i \in \mathbb{N}$, thus $\langle\langle h^i \rangle\rangle_G < M$. By assumption, we have $H < \langle\langle h^i \rangle\rangle_G$, hence $H < M$ and $M = N = H$. □

Lemma 12. *Let G be a group and let H, M be two subgroups of G such that $[G : M] < \infty$. Then $[H : (M \cap H)] \leq [G : M] < \infty$.*

Proof. Let $k := [G : M]$ and write

$$G = \bigsqcup_{i=1}^{k} M g_i.$$

Then, intersecting with H, we get

$$H = G \cap H = \bigsqcup_{i=1}^{k} (M g_i \cap H).$$

Fix $i \in \{1, \dots, k\}$. If $M g_i \cap H \neq \emptyset$, take any element $m g_i = h \in M g_i \cap H$. Then $M g_i \cap H = M m g_i \cap H = M h \cap H = M h \cap H h = (M \cap H) h$ and we are done. □

Lemma 13. *Let G be a group and $H < G$ a subgroup of finite index. Then*

$$\bigcap_{N \overset{f.i.}{\lhd} H} N = \bigcap_{N \overset{f.i.}{\lhd} G} N.$$

In particular, H is residually finite if and only if G is residually finite.

Proof.

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} H} N = \bigcap_{M \overset{\text{f.i.}}{<} H} M = \bigcap_{M \overset{\text{f.i.}}{<} G} M = \bigcap_{N \overset{\text{f.i.}}{\lhd} G} N,$$

where the first and third equalities follow from Lemma 10. The inclusion "$\supseteq$" in the second equality is obvious, whereas "$\subseteq$" in the second equality directly follows from Lemma 12. □

Proposition 14. *Let Γ be a $(2m, 2n)$-group such that any non-trivial normal subgroup of Γ has finite index. Let $H \lhd \Gamma$ be a non-trivial normal subgroup of Γ and assume that there is an element $h \in H$ such that $\langle\langle h^k \rangle\rangle_\Gamma > H$ for each $k \in \mathbb{N}$. Then H is a finitely presented torsion-free simple group.*

Proof. First note that by assumption H has finite index in Γ. By Lemma 11

$$H = \bigcap_{N \overset{\text{f.i.}}{\lhd} H} N$$

and hence by Lemma 13

$$H = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

In particular, Γ is non-residually finite and [16, Corollary 5.4] shows that H is simple. It is obvious that H is finitely presented and torsion-free, since it is a finite index subgroup of the finitely presented torsion-free group Γ. □

Corollary 15. *Let Γ be as in Example 1. Assume that there is an element $\gamma_0 \in \Gamma_0$ such that $\langle\langle \gamma_0^k \rangle\rangle_\Gamma = \Gamma_0$ for each $k \in \mathbb{N}$. Then Γ_0 is a finitely presented torsion-free simple group.*

Proof. This follows directly from Proposition 14 using Theorem 1(2). $\square$

One step towards the proof of Conjecture 1 (or an application of Corollary 15) could be the following proposition, whose detailed proof is given in Appendix A.4:

Proposition 16. *For Γ as defined in Example 1, we have $\langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma = \Gamma_0$ for each $k \in \mathbb{N}_0$.*

Remark. A calculation with MAGNUS ([49]) shows, that moreover $\langle\langle a_1^{12} \rangle\rangle_\Gamma = \langle\langle a_1^{24} \rangle\rangle_\Gamma = \Gamma_0$. See Table 2 for the orders of some quotients of Γ, illustrating that Conjecture 1 could be reasonable.

$\lvert\Gamma/\langle\langle w^k \rangle\rangle_\Gamma\rvert$	$k = 1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1$	2	4	2	4	2	4	2	4	2	4	2	4
a_2	2	4	2	4	2	4	2	4	2	4	2	4
a_3	2	4	2	4	2	4	2	4	2	4	2	4
b_1	2	4	2	4	2	4	2	4	2	4	2	4
b_2	2	4	2	4	2	4	2	4	2	4	2	4
b_3	2	4	2	4	2	4	2	4	2	4	2	4

Table 2: Order of $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$, $w \in \{a_1, a_2, a_3, b_1, b_2, b_3\}$, $k = 1, \ldots, 12$, in Example 1

In order to prove that Γ_0 has no proper finite index subgroups, it could be useful to have a non-trivial element $\gamma \in \Gamma$ such that for example γ and γ^2 are conjugate. But this is not possible by the following proposition which is a special case of a result of Bridson and Haefliger ([8]):

Proposition 17. *Let Γ be a $(2m, 2n)$-group and $\gamma \in \Gamma$ a non-trivial element. Then γ^k can only be conjugate to γ^l if $|k| = |l|$.*

Proof. Assume that γ^k and γ^l are conjugate for some $k, l \in \mathbb{Z}$. Then by [8, Proposition II.6.2(2)], γ^k and γ^l have the same translation length, and by [8, Theorem II.6.8(1)] we have $|k| = |l|$, using the fact that γ acts as a hyperbolic isometry on the CAT(0)-space $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. $\square$

The observation that Γ_0 has no subgroups of small index is reflected by the next proposition. We first give a lemma used in the proof of Proposition 19:

Lemma 18. *Let $w \in \Gamma_0$ be a (reduced) word of length 2, 4 or 6. Then $\langle\langle w \rangle\rangle_{\Gamma_0} = \Gamma_0$.*

Proof. We have used GAP ([28]). There are 84960 different elements to check. $\square$

Recall that we denote by $d(G)$ the minimal number of elements needed to generate the group G and by G^k the direct product of k copies of G.

Proposition 19. *$d(\Gamma_0^k) \leq 3$, if $k \leq 1230$.*

Proof. (Adapted from [68, Proof of Theorem 4.2]) There are 60 elements in Γ_0 of length 2 and 2400 elements of length 4. Since $w \neq w^{-1}$ and $|w| = |w^{-1}|$ for any non-trivial element $w \in \Gamma$, we can choose a subset $S = \{\gamma_1, \ldots, \gamma_{1230}\} \subset \Gamma_0$ satisfying $|S| = 1230$, $S \cap S^{-1} = \emptyset$, $|\gamma_i| = 2$ for $i = 1, \ldots, 30$ and $|\gamma_i| = 4$ for $i = 31, \ldots, 1230$. It follows that $|\gamma_{i_1}\gamma_{i_2}^{-1}| \in \{2, 4, 6\}$ whenever $\gamma_{i_1}, \gamma_{i_2}$ are different elements of S. Applying Lemma 18, we see that $\langle\langle \gamma_{i_1}\gamma_{i_2}^{-1} \rangle\rangle_{\Gamma_0} = \Gamma_0$. Note that Γ_0 is generated by two elements, for example by $\{a_1^2, b_2 b_1^{-1}\}$. For each $k \leq 1230$, we want to show that Γ_0^k is generated by the element $(\gamma_1, \ldots, \gamma_k)$ and the diagonal subgroup (which is for example generated by the two diagonal elements $(a_1^2,\dots,a_1^2)$ and $(b_2b_1^{-1},\dots,b_2b_1^{-1})$ in Γ_0^k). For $k=1$, this is obviously true. We assume that $2\le k\le 1230$ is fixed and that Γ_0^{k-1} is generated by the diagonal subgroup and $(\gamma_1,\dots,\gamma_{k-1})$. Let H be the subgroup of Γ_0^k generated by the diagonal subgroup and $(\gamma_1,\dots,\gamma_k)$. Our goal is to show that $H=\Gamma_0^k$. If we think Γ_0^{k-1} embedded in Γ_0^k as a subgroup $\Gamma_0^{k-1}\times\{1\}<\Gamma_0^{k-1}\times\Gamma_0=\Gamma_0^k$, then for any $\gamma\in\Gamma_0$ the group H contains by assumption $k-1$ elements of the form

$$(\gamma,1,\dots,1,*),\ \dots\ ,(1,\dots,1,\gamma,*),$$

where $*$ are certain elements in Γ_0 we do not have to care about. By construction, H contains also the element

$$(\gamma_1\gamma_k^{-1},\dots,\gamma_{k-1}\gamma_k^{-1},1)=(\gamma_1,\dots,\gamma_k)\cdot(\gamma_k^{-1},\dots,\gamma_k^{-1}).$$

Computing the commutators

$$[(\gamma,1,\dots,1,*),(\gamma_1\gamma_k^{-1},\dots,\gamma_{k-1}\gamma_k^{-1},1)],\ \dots\ ,[(1,\dots,1,\gamma,*),(\gamma_1\gamma_k^{-1},\dots,\gamma_{k-1}\gamma_k^{-1},1)],$$

we see that H contains the elements

$$([\gamma,\gamma_1\gamma_k^{-1}],1,\dots,1),\ \dots\ ,(1,\dots,1,[\gamma,\gamma_{k-1}\gamma_k^{-1}],1).$$

Let now N be the subgroup of Γ_0

$$N:=\langle[\gamma,\gamma_1\gamma_k^{-1}],\dots,[\gamma,\gamma_{k-1}\gamma_k^{-1}]:\gamma\in\Gamma_0\rangle<\Gamma_0.$$

Then N is a normal subgroup of Γ_0, since for each $g\in\Gamma_0$ and $j=1,\dots,k-1$

$$g[\gamma,\gamma_j\gamma_k^{-1}]g^{-1}=[g\gamma,\gamma_j\gamma_k^{-1}]\cdot[g,\gamma_j\gamma_k^{-1}]^{-1}\in N.$$

Note that by definition of N

$$\{\gamma_1\gamma_k^{-1}N,\dots,\gamma_{k-1}\gamma_k^{-1}N\}\subset Z(\Gamma_0/N).$$

Since $\langle\!\langle\gamma_j\gamma_k^{-1}\rangle\!\rangle_{\Gamma_0}=\Gamma_0$, we have $\langle\!\langle\gamma_j\gamma_k^{-1}N\rangle\!\rangle_{\Gamma_0/N}=\Gamma_0/N$ and $Z(\Gamma_0/N)=\Gamma_0/N$, i.e. Γ_0/N is abelian. But then $N=\Gamma_0$, because Γ_0 is perfect. In particular, Γ_0 is generated by the elements $[\gamma,\gamma_j\gamma_k^{-1}]$ and H contains therefore the j-th direct factor of Γ_0^k. Since

$$(1,\dots,1,\gamma)=(\gamma,\dots,\gamma)\cdot(\gamma^{-1},1,\dots,1)\cdot\ldots\cdot(1,\dots,1,\gamma^{-1},1),$$

H also contains the k-th direct factor of Γ_0^k, therefore $H=\Gamma_0^k$ and Γ_0^k is generated by three elements. $\square$

In most of our main examples (e.g. Example 1, 3, 6, 7, 8, 9, 10, 11, 14, 15 and 18) of Section 2 and Section 3 we always have $[\Gamma,\Gamma]=\Gamma_0$, where in addition Γ_0 is perfect. The next example is different in this regard (see also Appendix C.4).

Example 2.

$$R(3,3):=\left\{\begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_1, & a_1b_3a_1^{-1}b_3, \\ a_1b_2^{-1}a_2b_1^{-1}, & a_2b_1a_3^{-1}b_3^{-1}, & a_2b_2a_3^{-1}b_3, \\ a_2b_3a_3^{-1}b_2, & a_2b_3^{-1}a_3^{-1}b_1, & a_3b_1a_3b_2 \end{array}\right\}.$$

Theorem 2. (1) *The statements of Theorem 1 (1)-(5) also hold for* Γ.

(2) $[\Gamma,\Gamma]$ *is not perfect, of index* 32 *in* Γ *and* Γ_0 *is not perfect neither.*

Proof. (1) We use the same arguments as before, of course with different generators of P_h and P_v:

$$\begin{aligned}\rho_v(b_1) &= (1,5,4,3,2),\\ \rho_v(b_2) &= (2,6,5,4,3),\\ \rho_v(b_3) &= (2,3)(4,5),\\ \\ \rho_h(a_1) &= (1,5,6,2)(3,4),\\ \rho_h(a_2) &= (1,5,3)(2,6,4),\\ \rho_h(a_3) &= (1,3,5)(2,4,6).\end{aligned}$$

(2) It is easy to check that $[\Gamma,\Gamma]$ is the kernel of the surjective homomorphism given by

$$\begin{aligned}\Gamma &\to \mathbb{Z}_2^2\times\mathbb{Z}_8\\ a_1 &\mapsto (1,0,0)\\ a_2 &\mapsto (1,0,6)\\ a_3 &\mapsto (0,0,1)\\ b_1 &\mapsto (0,1,3)\\ b_2 &\mapsto (0,1,3)\\ b_3 &\mapsto (1,1,0).\end{aligned}$$

Note that the commutator subgroup of $[\Gamma,\Gamma]$ has index 6 in $[\Gamma,\Gamma]$ and that $\langle\!\langle a_1^2\rangle\!\rangle_\Gamma$ is a perfect normal subgroup of Γ of index 192. See Table 3 for the orders of some other quotients. Moreover, $[\Gamma_0,\Gamma_0]$ has index 64 in Γ, more precisely $\Gamma_0^{ab}\cong\mathbb{Z}_{16}$.

$\lvert\Gamma/\langle\!\langle w^k\rangle\!\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w=a_1$	48	192	48	192	48	192	48	192	48	192	48	192
a_2	8	16	24	32	8	48	8	64	24	16	8	96
a_3	4	24	4	48	4	24	4	96	4	24	4	48
b_1	4	8	12	16	4	24	4	32	12	8	4	48
b_2	4	8	12	16	4	24	4	32	12	8	4	48
b_3	16	96	16	192	16	96	16	192	16	96	16	192

Table 3: Order of $\Gamma/\langle\!\langle w^k\rangle\!\rangle_\Gamma$, $w\in\{a_1,a_2,a_3,b_1,b_2,b_3\}$, $k=1,\ldots,12$, in Example 2

□

Conjecture 2. Γ *is non-residually finite and*

$$\bigcap_{N\overset{f.i.}{\lhd}\Gamma} N = [[\Gamma,\Gamma],[\Gamma,\Gamma]] = \langle\!\langle a_1^{2k}\rangle\!\rangle_\Gamma,\ k\in\mathbb{N}.$$

Question 1. *Let* Γ *be a* $(2m,2n)$*-group such that any non-trivial normal subgroup of* Γ *has finite index. Assume that* $\Lambda\lhd\Gamma$ *is a perfect normal subgroup (of finite index). Is* Λ *simple?*

2.2 (A_6, P_v)-group, P_v primitive, not 2-transitive

Example 3.

$$R(3,5) := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_3^{-1}, & a_1 b_3 a_2^{-1} b_1, \\ a_1 b_4 a_2^{-1} b_5^{-1}, & a_1 b_5 a_2^{-1} b_5, & a_1 b_5^{-1} a_2^{-1} b_4^{-1}, \\ a_1 b_4^{-1} a_2 b_1^{-1}, & a_1 b_3^{-1} a_2^{-1} b_3, & a_1 b_2^{-1} a_2 b_4, \\ a_2 b_1 a_3^{-1} b_2, & a_2 b_2 a_3^{-1} b_1, & a_3 b_1 a_3 b_2, \\ a_3 b_3 a_3^{-1} b_3^{-1}, & a_3 b_4 a_3 b_4^{-1}, & a_3 b_5 a_3^{-1} b_5 \end{array} \right\}.$$

Theorem 3. (1) *$P_h = A_6$; $P_v \cong S_5 < S_{10}$ is primitive, not 2-transitive.*

(2) *There are two amalgam decompositions of Γ:*

$$F_5 *_{F_{25}} F_{13} \cong \Gamma \cong F_3 *_{F_{21}} F_{11}.$$

There is a vertical decomposition of Γ_0

$$\Gamma_0 \cong F_9 *_{F_{49}} F_9,$$

acting locally like A_6 (but possibly not effectively) on $\mathcal{T}_6$, and a horizontal decomposition

$$\Gamma_0 \cong F_5 *_{F_{41}} F_5 < \mathrm{Aut}(\mathcal{T}_{10}),$$

where the (effective) action on $\mathcal{T}_{10}$ is locally like $S_5 < S_{10}$, in particular locally primitive, but not locally 2-transitive.

(3) *$H_b^2(\Gamma; \mathbb{R})$ is infinite dimensional as $\mathbb{R}$-vector space (cf. Theorem 1(5)).*

(4) *Γ is SQ-universal, in particular not virtually simple.*

(5) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(6) *Γ is not linear over any field.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (1,5,4,3,2), \\ \rho_v(b_2) &= (2,6,5,4,3), \\ \rho_v(b_3) &= (1,2)(5,6), \\ \rho_v(b_4) &= (1,2,6,5)(3,4), \\ \rho_v(b_5) &= (1,2)(5,6), \\ \\ \rho_v(a_1) &= (1,7,9,10,3,2)(4,6,5), \\ \rho_v(a_2) &= (1,8,9)(2,4,10)(5,6,7), \\ \rho_v(a_3) &= (1,9)(2,10)(5,6). \end{aligned}$$

The action of $P_v^{(2)}$ on $S(x_v, 2)$ has two orbits of size 60 and 30 respectively. Observe that in general the action of $P_v^{(2)}$ on $S(x_v, 2)$ is transitive if and only if P_v is a 2-transitive permutation group. Note that P_v acts like S_5 on the set of 2-subsets of $\{1,2,3,4,5\}$.

(2) Again, use [69, Theorem I1.18]. The explicit horizontal decomposition of Γ_0 can be found in Appendix A.5.

(3) Observe that in the horizontal decomposition $\Gamma \cong F_3 *_{F_{21}} F_{11}$ we have $|F_{21}\backslash F_3/F_{21}| = 3$ (for a computation of this expression, see Proposition 22) and $|F_{11}/F_{21}| = 2$. Now apply [27, Theorem 1.1] which states that $H_b^2(A *_C B; \mathbb{R})$ is an infinite dimensional $\mathbb{R}$-vector space if $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$. Note that the assumptions of this theorem are not fulfilled in the two $(F_3 *_{F_{13}} F_7)$-decompositions of Example 1, since $|F_{13}\backslash F_3/F_{13}| = 2$ due to the 2-transitivity of P_h and P_v in Example 1.

(4) Recall that a countable group G is called *SQ-universal*, if every countable group can be embedded in a quotient of G. By a result of Ilya Rips, mentioned in [2, Chapter 9.15], an amalgam $A *_C B$ is SQ-universal provided that $C \neq B$ and $|C\backslash A/C| \geq 3$. We apply this to $\Gamma \cong F_3 *_{F_{21}} F_{11}$. Note that Γ does not satisfy the assumptions of the normal subgroup theorem [16, Theorem 4.1], since H_2 is not locally 2-transitive and hence not locally ∞-transitive.

(5) This is a short computation.

(6) It follows from Proposition 38 in Section 4.9.

□

Proposition 20. *Let Γ be as in Example 3. Then $\langle\langle a_1^k \rangle\rangle_\Gamma = \Gamma_0$ for $k \in \{2+6l, 4+6l\}$, $l \in \mathbb{N}_0$. Moreover, $\langle\langle a_1^6 \rangle\rangle_\Gamma = \langle\langle a_1^{12} \rangle\rangle_\Gamma = \langle\langle a_1^{18} \rangle\rangle_\Gamma = \Gamma_0$.*

Proof. For the first part, we only give the idea of the proof, which is essentially the same as in the proof of Proposition 16. First show that $\langle\langle b_4 b_5 \rangle\rangle_\Gamma = \Gamma_0$ and $\langle\langle b_5^2 \rangle\rangle_\Gamma = \Gamma_0$, then show that for $l \in \mathbb{N}_0$

$$a_1^{-k}(b_5^{-1} b_3 a_1^k b_3^{-1} b_5) = \begin{cases} b_4 b_5, & k = 2+6l \\ b_5^2, & k = 4+6l. \end{cases}$$

We have checked the second part with MAGNUS ([49]). □

Conjecture 3. *The group Γ of Example 3 is non-residually finite. More precisely*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = \Gamma_0.$$

See Table 4 for the orders of some quotients (it looks like Table 2).

$\lvert\Gamma/\langle\langle w^k \rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1$	2	4	2	4	2	4	2	4	2	4	2	4
a_2	2	4	2	4	2	4	2	4	2	4	2	4
a_3	2	4	2	4	2	4	2	4	2	4	2	4
b_1	2	4	2	4	2	4	2	4	2	4	2	4
b_2	2	4	2	4	2	4	2	4	2	4	2	4
b_3	2	4	2	4	2	4	2	4	2	4	2	4
b_4	2	4	2	4	2	4	2	4	2	4	2	4
b_5	2	4	2	4	2	4	2	4	2	4	2	4

Table 4: Order of $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$, $w \in \{a_1, a_2, a_3, b_1, \ldots, b_5\}$, $k = 1, \ldots, 12$, in Example 3

Remark. We construct in Example 18 a $(10,10)$-group Γ such that the corresponding subgroup Γ_0 is not simple and

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N = \Gamma_0.$$

However, in Example 18, P_h is transitive but not primitive.

Conjecture 4. *Every orbit of the $H_2(x_v)$-action on $\partial_\infty \mathcal{T}_{10}$ is uncountable.*

"Proof". Studying the orbits of the local action of H_2 on finite spheres $S(x_v, k)$, we believe that the orbit of any boundary point $\omega \in \partial_\infty \mathcal{T}_{10}$ under the $H_2(x_v)$-action contains the uncountable boundary at infinity $\partial_\infty \mathcal{T}_{10;4,7}$ of a certain infinite subtree $\mathcal{T}_{10;4,7} \subset \mathcal{T}_{10}$. This subtree contains $S(x_v, 1)$ and the degree of a vertex $y_v \neq x_v$ is either 4 or 7 (depending on ω), but constant on finite spheres $S(x_v, k)$.

More precisely, we fix as usual a vertex $x_v \in \mathcal{T}_{10}$ and imagine paths (without backtracking) in $\mathcal{T}_{10}$ originating at x_v to be labelled by (reduced) words in $\langle b_1, \ldots, b_5 \rangle$. Using the explicit isomorphism $E_v \cong \{1, \ldots, 10\}$ described in Section 1, we identify the sphere $S(x_v, k)$ with the set of k-tuples

$$\{(e_1, \ldots, e_k) \in \{1, \ldots, 10\}^k : e_i + e_{i+1} \neq 11, \ \forall i \in \{1, \ldots, k-1\}\}.$$

For each $k \geq 2$, we define an equivalence relation $\sim_k$ on $S(x_v, k)$ as follows. First, $\sim_2$ gives a partition of $S(x_v, 2)$ into two equivalence classes consisting of 30 and 60 elements respectively. The equivalence class with 30 elements is $\{(1,3)$, $(1,5)$, $(1,9)$, $(2,6)$, $(2,7)$, $(2,10)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,1)$, $(4,4)$, $(4,9)$, $(5,2)$, $(5,8)$, $(5,9)$, $(6,1)$, $(6,8)$, $(6,10)$, $(7,3)$, $(7,7)$, $(7,8)$, $(8,2)$, $(8,4)$, $(8,10)$, $(9,1)$, $(9,3)$, $(9,6)$, $(10,2)$, $(10,5)$, $(10,7)\}$. For $k \geq 3$ we define

$$(e_1, \ldots, e_k) \sim_k (f_1, \ldots, f_k) :\Longleftrightarrow (e_i, e_{i+1}) \sim_2 (f_i, f_{i+1}) \ \forall i \in \{1, \ldots, k-1\}.$$

Note that we have 2^{k-1} equivalence classes on $S(x_v, k)$ with respect to $\sim_k$, where the number of elements in each class is $10 \cdot 6^j \cdot 3^{k-1-j}$ for a $j \in \{0, \ldots, k-1\}$. We have checked that the $H_2(x_v)$-action induces exactly $\sim_k$ on $S(x_v, k)$ for $k = 2, 3, 4$. $\square$

As a "corollary" of Conjecture 4, we have

Conjecture 5. $\mathrm{QZ}(H_2) = 1$.

"Proof". If Conjecture 4 holds, then we follow verbatim the proof of [15, Proposition 3.1.2, 1)]: Let $S \subset \partial_\infty \mathcal{T}_{10}$ be the set of fixed points of hyperbolic elements in $\mathrm{QZ}(H_2)$. Then S is countable, since $\mathrm{QZ}(H_2)$ is countable, which follows directly from the fact that $\mathrm{QZ}(H_2)$ is discrete (see [15, Proposition 1.2.1, 2)]). Moreover, S is H_2-invariant, since $\mathrm{QZ}(H_2)$ is a normal subgroup of H_2. We conclude by Conjecture 4 that S is empty, in other words $\mathrm{QZ}(H_2)$ has no hyperbolic elements. On the other hand, $\mathrm{QZ}(H_2)$ acts by [15, Proposition 1.2.1, 2)] freely on the vertices of $\mathcal{T}_{10}$ (in particular, there are no elliptic elements in $\mathrm{QZ}(H_2) \setminus \{1\}$), hence $|\mathrm{QZ}(H_2)| \leq 2$. But then, $\mathrm{QZ}(H_2) \subseteq Z(H_2) = 1$. $\square$

See the following table to check that small powers of $b_1, \ldots, b_5$ are not in $\Lambda_2 < \mathrm{QZ}(H_2)$ (see also Appendix A.6 for all words w of length 2). For instance, it follows that $b_1^j \notin \Lambda_2$ if $1 \leq j < 3000$ by the following Lemma.

Lemma 21. *Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n) \rangle$ be a $(2m, 2n)$-group and let $b \in \langle b_1, \ldots, b_n \rangle$ such that $b^j \in \Lambda_2$ for some $j \in \mathbb{N}$. Then $|\rho_v^{(k)}(b)| \leq j$ for each $k \in \mathbb{N}$.*

Proof. Fix any $k \in \mathbb{N}$. Using the identification

$$\Lambda_2 \cong \bigcap_{k \in \mathbb{N}} \ker \rho_v^{(k)}$$

we get

$$\left(\rho_v^{(k)}(b)\right)^j = \rho_v^{(k)}(b^j) = 1_{\mathrm{Sym}(E_h^{(k)})}$$

hence $|\rho_v^{(k)}(b)| \le j$. $\square$

$\rho_v^{(k)}(w)$	$k=1$	2	3	4	5
$w = b_1$	5	10	100	600	3000
b_2	5	10	100	600	3000
b_3	2	10	50	100	1000
b_4	4	8	40	200	1000
b_5	2	4	20	40	1200

Compare this to the table below, where we already know that $\mathrm{QZ}(H_1) = 1$ (by [15, Proposition 3.1.2, 1)]).

$\rho_h^{(k)}(w)$	$k=1$	2	3	4
$w = a_1$	6	12	72	432
a_2	3	6	12	72
a_3	2	4	8	16

Conjecture 5 implies another conjecture:

Conjecture 6. *Let $N \lhd \Gamma$ be a non-trivial normal subgroup of infinite index. Then Γ/N is an infinite group having property (T) of Kazhdan.*

"Proof". We know that $\mathrm{QZ}(H_1) = 1$ (see [15, Proposition 3.1.2, 1)]) and assume that $\mathrm{QZ}(H_2) = 1$ (see Conjecture 5). For $1 \neq N \lhd \Gamma$ and $i = 1, 2$, we have $1 \neq \overline{\mathrm{pr}_i(N)} \lhd H_i$. By [15, Proposition 1.2.1] $H_i/\overline{\mathrm{pr}_i(N)}$ is compact. We can apply [16, Proposition 3.1] to conclude that Γ/N has property (T). Note that there are uncountably many non-isomorphic infinite quotients Γ/N, since Γ is SQ-universal (see [56], the proof is based on the fact that there are uncountably many non-isomorphic finitely generated groups, but each quotient Γ/N, being countable, has only countably many finitely generated subgroups). $\square$

2.2.1 Double cosets

Proposition 22. *Let Γ be a $(2m, 2n)$-group. Suppose that $P_h < S_{2m}$ is transitive. Then there is a bijection between the set of orbits of the diagonal action of P_h on $\{1, \ldots, 2m\} \times \{1, \ldots, 2m\}$ and the set $F_{1-2m+2mn} \backslash F_n / F_{1-2m+2mn} = \{F_{1-2m+2mn} f F_{1-2m+2mn} \mid f \in F_n\}$ of double cosets, where*

$$\Gamma \cong F_n *_{F_{1-2m+2mn}} F_{1-m+mn}$$

is the vertical decomposition given by Proposition 2(1a). In particular, $|F_{1-2m+2mn} \backslash F_n / F_{1-2m+2mn}|$ is the rank *of P_h (in the terminology of [24, p.67]) and can be easily computed without explicitly knowing the explicit amalgam decomposition, for example using the GAP-command*

```
1 + Size(OrbitLengths(Ph, Arrangements([1..2*m],2),OnTuples));
```

Another consequence is that

$$|F_{1-2m+2mn} \backslash F_n / F_{1-2m+2mn}| = 2,$$

if and only if P_h is a 2-transitive permutation group. (As always, similar statements can be made for P_v.)

Proof. We define $B := F_n$, $C := F_{1-2m+2mn}$. Let $\mathcal{T}'_{2m}$ be the Bass-Serre tree on which the amalgam $\Gamma \cong B *_C F_{1-m+mn}$ naturally acts and let x_h be the vertex of $\mathcal{T}'_{2m}$ such that $B = \mathrm{Stab}_\Gamma(x_h)$. Denote by Ω the set of edges in $\mathcal{T}'_{2m}$ originating from x_h and let $\omega \in \Omega$ be the edge such that $\mathrm{Stab}_\Gamma(\omega) = C$. Note that $|\Omega| = [B : C] = [F_n : F_{1-2m+2mn}] = 2m$. By construction, the action of P_h on $\{1, \dots, 2m\} \cong E_h$ is equivalent (permutation isomorphic) to the action of B on Ω. We want to define a bijection

$$\varphi : \{\text{Orbits of } B \curvearrowright \Omega \times \Omega\} \longrightarrow C \backslash B / C.$$

Let $(\omega_1, \omega_2) \in \Omega \times \Omega$. We denote by $[(\omega_1, \omega_2)]$ its B-orbit under the diagonal action, in particular $[(\omega_1, \omega_2)] = [(b\omega_1, b\omega_2)]$ for each $b \in B$. Since B acts transitively on Ω, we can choose $b_1, b_2 \in B$ such that $\omega = b_1\omega_1 = b_2\omega_2$. Now we define

$$\varphi\left([(\omega_1, \omega_2)]\right) := Cb_1b_2^{-1}C \in C \backslash B / C.$$

We first show that φ is independent of the choice of b_1, b_2. Take $\tilde{b}_1, \tilde{b}_2 \in B$ such that $\omega = \tilde{b}_1\omega_1 = \tilde{b}_2\omega_2$. Then $b_i\tilde{b}_i^{-1}\omega = b_i\omega_i = \omega$, $(i = 1, 2)$, hence $b_i\tilde{b}_i^{-1} \in C$, i.e. $Cb_1 = C\tilde{b}_1$ and $b_2^{-1}C = \tilde{b}_2^{-1}C$ which implies $C\tilde{b}_1\tilde{b}_2^{-1}C = Cb_1b_2^{-1}C$. Next we show that φ is independent of the representative of $[(\omega_1, \omega_2)]$. Any representative of $[(\omega_1, \omega_2)]$ has the form $(b\omega_1, b\omega_2)$ for some $b \in B$. But then $\omega = b_1b^{-1}(b\omega_1) = b_2b^{-1}(b\omega_2)$ and

$$\varphi\left([(b\omega_1, b\omega_2)]\right) = Cb_1b^{-1}(b_2b^{-1})^{-1}C = Cb_1b_2^{-1}C.$$

This proves that φ is well-defined.
Note that $\varphi([(\omega, b\omega)]) = CbC$ for each $b \in B$, hence φ is surjective.
To show the injectivity of φ, assume that

$$\varphi\left([(\omega_1, \omega_2)]\right) = Cb_1b_2^{-1}C = C\tilde{b}_1\tilde{b}_2^{-1}C = \varphi\left([(\tilde{\omega}_1, \tilde{\omega}_2)]\right),$$

such that $\omega = b_1\omega_1 = b_2\omega_2 = \tilde{b}_1\tilde{\omega}_1 = \tilde{b}_2\tilde{\omega}_2$. The assumption $Cb_1b_2^{-1}C = C\tilde{b}_1\tilde{b}_2^{-1}C$ implies that there is some $c \in C$ such that

$$\begin{aligned} cb_1b_2^{-1} &\in \tilde{b}_1\tilde{b}_2^{-1}C, \\ \tilde{b}_2\tilde{b}_1^{-1}cb_1b_2^{-1} &\in C, \\ \tilde{b}_2\tilde{b}_1^{-1}cb_1b_2^{-1}\omega &= \omega, \\ cb_1b_2^{-1}\omega &= \tilde{b}_1\tilde{b}_2^{-1}\omega, \end{aligned}$$

hence

$$[(\omega_1, \omega_2)] = [(\omega, b_1b_2^{-1}\omega)] = [(c\omega, cb_1b_2^{-1}\omega)] = [(\omega, \tilde{b}_1\tilde{b}_2^{-1}\omega)] = [(\tilde{\omega}_1, \tilde{\omega}_2)].$$

$\square$

2.2.2 Blocking pairs

One method to prove the SQ-universality of an amalgamated free product is a criterion of Paul Schupp ([63]) using the notion of a blocking pair.

Definition. (see [63]) Let $C < A$ be groups. A pair $\{x_1, x_2\}$ of distinct elements in $A \setminus C$ is called a *blocking pair* for $C < A$ if

i) $x_i^\epsilon x_j^\delta \notin C \setminus \{1\}$, for all $i, j = 1, 2$; $\epsilon, \delta = \pm 1$.

ii) $x_i^\epsilon c x_j^\delta \notin C$, if $c \in C \setminus \{1\}$; $i, j = 1, 2$; $\epsilon, \delta = \pm 1$.

Proposition 23. *(1) (see [63]) If there is a blocking pair for $C < A$ or a blocking pair for $C < B$, then the amalgam $A *_C B$ is SQ-universal.*

(2) If there is a blocking pair for $C < A$, then $|C \backslash A/C| \geq 3$.

(3) Let Γ be a $(2m, 2n)$-group. Suppose that $P_h < S_{2m}$ is transitive. Then there is no blocking pair for $C < B$ and no blocking pair for $C < A$, where

$$B *_C A := F_n *_{F_{1-2m+2mn}} F_{1-m+mn} \cong \Gamma$$

is the vertical decomposition given by Proposition 2(1a).

Proof. (1) See [63] (using small cancellation theory).

(2) Let $\{x_1, x_2\}$ be a blocking pair for $C < A$. Obviously, we have $Cx_1C \neq C \neq Cx_2C$. Assume that $Cx_1C = Cx_2C$, thus there exist $c_1, c_2 \in C$ such that $x_1 = c_1x_2c_2$. If $c_1 = 1 = c_2$, then $x_1 = x_2$, a contradiction. If $c_1 \neq 1$, then $x_1^{-1}c_1x_2 = c_2^{-1} \in C$, a contradiction. If $c_2 \neq 1$, then $x_2c_2x_1^{-1} = c_1^{-1} \in C$, again a contradiction to the blocking pair assumption.

(3) By (2), there is no blocking pair for $C < A$, since $|C \backslash A/C| \leq |A/C| = 2 < 3$. Let x_1 be in a blocking pair for $C < B$. Let $b \neq 1$ be in $\ker(\rho_v : \langle b_1, \ldots, b_n \rangle \twoheadrightarrow P_h)$. Since $[B : C] = 2m < \infty$, there is an integer $k \in \mathbb{N}$ such that $b^k \in C$. Let $c := b^k$, then $c \in \ker\rho_v \setminus \{1\}$ fixes the 1-sphere around the vertex "B" in the corresponding Bass-Serre tree (see Figure 5), in particular c fixes the edge "Cx_1", hence $Cx_1c = Cx_1$, but then

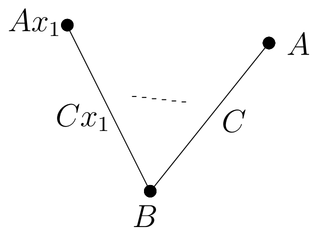


Figure 5: Illustration in the proof of Proposition 23(3)

$x_1cx_1^{-1} \in C$ is a contradiction to the assumption that x_1 is in a blocking pair for $C < B$. □

2.2.3 A homomorphism due to Bernhard H. Neumann

Proposition 24. *(cf. [55]) Let A, B, C be groups, $i_A : C \to A$ and $i_B : C \to B$ injective homomorphisms and assume that $A \neq 1$. Via the identifications $C \cong i_A(C)$, $C \cong i_B(C)$, we think* *of C as a subgroup of A and B, and we write as usual $A *_C B$ to denote the amalgamated free product*

$$\langle A * B \mid i_A(c) = i_B(c),\, c \in C\rangle.$$

Then there is a surjective homomorphism

$$\rho : A *_C B \twoheadrightarrow P < \mathrm{Sym}(A \times B),$$

such that $P \neq 1$. In particular, if ρ is not injective, we get a non-trivial proper quotient $P \cong (A *_C B)/\ker\rho$ of $A *_C B$ (if ρ is injective, then $P \cong A *_C B$).

Proof. (cf. [55]) We fix right coset representatives $S_A = \{a_1 := 1, a_2, a_3, \ldots\}$, $S_B = \{b_1 := 1, b_2, b_3, \ldots\}$ of C in A and B respectively, i.e. $A = \sqcup_i Ca_i$, $B = \sqcup_j Cb_j$. We will define two homomorphisms $\rho_A : A \to \mathrm{Sym}(A \times B)$ and $\rho_B : B \to \mathrm{Sym}(A \times B)$ as follows. Let $(x, y) \in A \times B$, then $\rho_A(a)(x, y) := (ax, y)$. Obviously, ρ_A is a homomorphism:

$$\rho_A(a\tilde{a})(x, y) = (a\tilde{a}x, y) = \rho_A(a)(\tilde{a}x, y) = \rho_A(a)\rho_A(\tilde{a})(x, y).$$

To define $\rho_B(b)(x, y)$, note that with respect to the chosen (fixed) right coset representatives, we have unique decompositions

$$x = c_x a_x,\; y = c_y b_y,\; bc_x b_y = c_z b_z \quad (c_x, c_y, c_z \in C,\; a_x \in S_A,\; b_y, b_z \in S_B).$$

Now we define $\rho_B(b)(x, y) := (c_z a_x, c_y b_z)$. We check that ρ_B is a homomorphism:

$$\rho_B(b\tilde{b})(x, y) = (c_t a_x, c_y b_t),$$

where $b\tilde{b}c_x b_y = c_t b_t$ is the unique decomposition ($c_t \in C$, $b_t \in S_B$).

$$\rho_B(\tilde{b})(x, y) = (c_r a_x, c_y b_r),$$

where $\tilde{b}c_x b_y = c_r b_r$ is the unique decomposition ($c_r \in C$, $b_r \in S_B$). Hence,

$$\rho_B(b)\rho_B(\tilde{b})(x, y) = \rho_B(b)(c_r a_x, c_y b_r) = (c_t a_x, c_y b_t) = \rho_B(b\tilde{b})(x, y),$$

since $bc_r b_r = b\tilde{b}c_x b_y = c_t b_t$.

Let $c \in C$, then

$$\rho_B(c)(x, y) = (cc_x a_x, c_y b_y) = (cx, y) = \rho_A(c)(x, y),$$

in other words, $\rho_A \circ i_A = \rho_B \circ i_B$. By the universal property of $A *_C B$, the desired homomorphism $\rho : A *_C B \twoheadrightarrow P$ exists (see the following diagram), where $P < \mathrm{Sym}(A \times B)$ is generated by $\{\rho_A(A), \rho_B(B)\} \subseteq \mathrm{Sym}(A \times B)$. Obviously, $P \neq 1$, since $A \neq 1$ (by assumption) and $\rho_A(a)(1_A, 1_B) = (a, 1_B)$.

$$\begin{tikzcd}
C \arrow[r, "i_B"] \arrow[d, "i_A"'] & B \arrow[d] \arrow[ddr, bend left, "\rho_B"] & \\
A \arrow[r] \arrow[drr, bend right, "\rho_A"'] & A *_C B \arrow[dr, dotted, "\rho"] & \\
 & & P
\end{tikzcd}$$

$\square$

Question 2. *Let Γ be as defined in Example 3. Is there an amalgam decomposition $A *_C B$ of Γ (or Γ_0) such that ρ is not injective?*

2.2.4 A result of Roger C. Lyndon

Perhaps useful in the construction of infinite quotients of amalgamated free products could be the following proposition of Roger C. Lyndon:

Proposition 25. *(see [46, Proposition 1.3]) Let $G = A *_C B$ be an amalgamated free product. Let $N_A \lhd A$, $N_B \lhd B$ be normal subgroups such that $N_A \cap C = N_B \cap C$. Then*

$$G/N \cong A/N_A *_{C/N_C} B/N_B,$$

where $N_C := N_A \cap C = N_B \cap C$ and $N := \langle\langle N_A \cup N_B \rangle\rangle_G$.

Proof. See [46] or [21]. □

2.2.5 Embedding in (A_{10}, A_{12})-group

We now embed our $(A_6, S_5 < S_{10})$-group Γ of Example 3 in a (A_{10}, A_{12})-group. This group will satisfy the assumptions of the normal subgroup theorem ([16, Theorem 4.1] or Proposition 9). If Conjecture 3 is true, then also the (A_{10}, A_{12})-group is non-residually finite and by [16, Corollary 5.4] virtually simple. We mention that the same ideas will indeed lead in Section 3 to the construction of virtually simple (A_6, A_{16})-, (A_8, A_{14})-, $(\mathrm{ASL}_3(2)), A_{14})$- and (A_{10}, A_{10})-groups!

Example 4.

$$R(5,6) := \left\{ \begin{array}{lllll} \underline{a_1 b_1 a_1^{-1} b_2^{-1}}, & \underline{a_1 b_2 a_2^{-1} b_3^{-1}}, & \underline{a_1 b_3 a_2^{-1} b_1}, & \underline{a_1 b_4 a_2^{-1} b_5^{-1}}, & \underline{a_1 b_5 a_2^{-1} b_5}, \\ a_1 b_6 a_5^{-1} b_6, & a_1 b_6^{-1} a_4 b_6^{-1}, & \underline{a_1 b_5^{-1} a_2^{-1} b_4^{-1}}, & \underline{a_1 b_4^{-1} a_2 b_1^{-1}}, & \underline{a_1 b_3^{-1} a_2^{-1} b_3}, \\ \underline{a_1 b_2^{-1} a_2 b_4}, & \underline{a_2 b_1 a_3^{-1} b_2}, & \underline{a_2 b_2 a_3^{-1} b_1}, & a_2 b_6 a_2^{-1} b_6^{-1}, & \underline{a_3 b_1 a_3 b_2}, \\ \underline{a_3 b_3 a_3^{-1} b_3^{-1}}, & \underline{a_3 b_4 a_3 b_4^{-1}}, & \underline{a_3 b_5 a_3^{-1} b_5}, & a_3 b_6 a_3^{-1} b_6, & a_4 b_1 a_5 b_5^{-1}, \\ a_4 b_2 a_5 b_1, & a_4 b_3 a_5^{-1} b_4, & a_4 b_4 a_4^{-1} b_3, & a_4 b_5 a_5 b_4^{-1}, & a_4 b_6 a_5 b_2, \\ a_4 b_5^{-1} a_5 b_6, & a_4 b_3^{-1} a_5 b_1^{-1}, & a_4 b_2^{-1} a_5 b_5, & a_4 b_1^{-1} a_5 b_2^{-1}, & a_5 b_3 a_5 b_4 \end{array} \right\}.$$

Theorem 4. (1) *$P_h = A_{10}$, $P_v = A_{12}$.*

(2) *Γ is irreducible.*

(3) *The $(A_6, S_5 < S_{10})$-group of Example 3 injects in Γ.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (1,9,8,3,2)(4,6)(5,7),\\
\rho_v(b_2) &= (2,10,9,8,3)(4,6)(5,7),\\
\rho_v(b_3) &= (1,2)(5,7,6)(9,10),\\
\rho_v(b_4) &= (1,2,10,9)(3,8)(4,6,5),\\
\rho_v(b_5) &= (1,2)(4,6)(5,7)(9,10),\\
\rho_v(b_6) &= (1,7,5)(4,10,6),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,9,11,12,3,2)(4,8,5)(6,7),\\
\rho_h(a_2) &= (1,10,11)(2,4,12)(5,8,9),\\
\rho_h(a_3) &= (1,11)(2,12)(5,8)(6,7),\\
\rho_h(a_4) &= (1,10,4,5)(2,12)(3,9)(6,7,8,11),\\
\rho_h(a_5) &= (1,11)(2,5,6,7)(3,12,8,9)(4,10).
\end{aligned}$$

(2) We compute $|P_h^{(2)}| = 1814400 \cdot 181440^{10}$ and apply now Proposition 1(1a).

(3) The $(A_6, S_5 < S_{10})$-complex of Example 3 embeds in X inducing a π_1-injection by Proposition 5(1). Observe that the underlined 15 elements in $R(5,6)$ are identical with the 15 relators of $R(3,5)$ in Example 3.

$\square$

2.2.6 Two different amalgam decompositions

The next example gives two amalgam decompositions of a Γ_0, both of the form $F_9 *_{F_{81}} F_9$, but acting substantially differently on the corresponding trees $\mathcal{T}_{2m} = \mathcal{T}_{10}$ and $\mathcal{T}_{2n} = \mathcal{T}_{10}$.

Example 5.

$$R(5,5) := \left\{ \begin{array}{lllll}
a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_1, & a_1b_4a_2^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_5,\\
a_1b_5^{-1}a_2^{-1}b_4^{-1}, & a_1b_4^{-1}a_2b_1^{-1}, & a_1b_3^{-1}a_3^{-1}b_3, & a_1b_2^{-1}a_2b_4, & a_2b_1a_2b_2,\\
a_2b_3a_4^{-1}b_3^{-1}, & a_3b_1a_3^{-1}b_1^{-1}, & a_3b_2a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3^{-1}b_5^{-1},\\
a_3b_3^{-1}a_5^{-1}b_3, & a_4b_1a_4^{-1}b_2, & a_4b_2a_4^{-1}b_1, & a_4b_3a_5^{-1}b_3^{-1}, & a_4b_4a_4b_4^{-1},\\
a_4b_5a_4^{-1}b_5, & a_5b_1a_5^{-1}b_1^{-1}, & a_5b_2a_5^{-1}b_2^{-1}, & a_5b_4a_5^{-1}b_4^{-1}, & a_5b_5a_5^{-1}b_5^{-1}
\end{array} \right\}.$$

Theorem 5. (1) *$P_h = A_{10}$; $P_v \cong S_5 < S_{10}$ is primitive, not 2-transitive.*

(2) *Γ is irreducible.*

(3) *Γ_0 splits as $F_9 *_{F_{81}} F_9$ in two different ways, one acting locally 2-transitively on the first tree $\mathcal{T}_{2m} = \mathcal{T}_{10}$ (in fact locally transitively on the boundary at infinity $\partial_\infty \mathcal{T}_{10}$), the other acting locally primitively, but not locally 2-transitively on the second tree $\mathcal{T}_{2n} = \mathcal{T}_{10}$ (in particular not locally transitively on $\partial_\infty \mathcal{T}_{10}$).*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (1,9,2),\\
\rho_v(b_2) &= (2,10,9),\\
\rho_v(b_3) &= (1,2,4,5,3)(6,8,10,9,7),\\
\rho_v(b_4) &= (1,2,10,9)(4,7),\\
\rho_v(b_5) &= (1,2)(9,10),\\
\\
\rho_h(a_1) &= (1,7,9,10,3,2)(4,6,5),\\
\rho_h(a_2) &= (1,8,9)(2,4,10)(5,6,7),\\
\rho_h(a_3) &= (),\\
\rho_h(a_4) &= (1,9)(2,10)(5,6),\\
\rho_h(a_5) &= ().
\end{aligned}$$

(2) Again, this follows from $|P_h^{(2)}| = 1814400 \cdot 181440^{10}$.

(3) Use the same ideas as in the preceding decompositions.

□

2.3 (A_6, M_{12})-group

Example 6.

$$R(3,6) := \left\{ \begin{array}{lll}
a_1b_1a_2^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_1^{-1}, & a_1b_3a_1^{-1}b_3^{-1},\\
a_1b_4a_1^{-1}b_4^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1},\\
a_1b_1^{-1}a_2b_2, & a_2b_1a_2b_3^{-1}, & a_2b_3a_2b_4^{-1},\\
a_2b_4a_3^{-1}b_5^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1},\\
a_2b_5^{-1}a_3b_4, & a_3b_1a_3^{-1}b_2^{-1}, & a_3b_2a_3^{-1}b_1^{-1},\\
a_3b_3a_3b_6^{-1}, & a_3b_5a_3^{-1}b_4^{-1}, & a_3b_6a_3b_3^{-1}
\end{array} \right\}.$$

Theorem 6. (1) *$P_h = A_6$, $P_v \cong M_{12}$ (Mathieu group).*

(2) *Any non-trivial normal subgroup of Γ has finite index.*

(3) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (2,6,5),\\
\rho_v(b_2) &= (1,2,5),\\
\rho_v(b_3) &= (2,5)(3,4),\\
\rho_v(b_4) &= (2,5,4),\\
\rho_v(b_5) &= (2,3,5),\\
\rho_v(b_6) &= (2,5)(3,4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,2)(5,6)(7,8)(11,12),\\
\rho_h(a_2) &= (1,2,7,5,4,3)(6,11,12,10,9,8),\\
\rho_h(a_3) &= (1,2)(3,6)(4,5)(7,10)(8,9)(11,12).
\end{aligned}$$

Observe that M_{12} is already generated by $\rho_h(a_1) =: \sigma$ and $\rho_h(a_2) =: \tau$, since $\rho_h(a_3) = \sigma\tau^3\sigma\tau\sigma\tau^2\sigma\tau^2\sigma\tau\sigma\tau^3\sigma$. As a by-product, we get the following short presentation of M_{12} with two generators:

$$M_{12} \cong \langle \sigma, \tau \mid \sigma^2, \tau^6, (\sigma\tau)^5, (\sigma\tau\sigma\tau^5)^4, (\sigma\tau^2)^6, (\sigma\tau\sigma\tau^4)^5 \rangle.$$

(2) We apply Proposition 9 or [16, Corollary 5.3], using the fact that

$$\mathrm{Stab}_{P_v}(\{1\}) = \langle (2,8,10,12,5)(3,4,7,6,9), (2,3,6,9)(5,10,7,12), (5,8)(6,7)(9,10)(11,12) \rangle \cong M_{11}$$

is a non-abelian simple group.

(3) This is a short computation.

□

Conjecture 7. Γ_0 *is a simple group.*

Remark. Analyzing many $(4,12)$-groups, we see that $P_v \cong M_{12}$ can be generated in several ways by $\{\rho_h(a_1), \rho_h(a_2)\}$. We have found seven different cycle structures for $\{\rho_h(a_1), \rho_h(a_2)\}$ generating M_{12}. They are listed in Table 5:

$\rho_h(a_1)$	$\rho_h(a_2)$
$(3,4)(5,6)(7,8)(9,10)$	$(1,7,5,3,2)(6,12,11,10,8)$
$(3,4)(5,6)(7,8)(9,10)$	$(1,6,5,9,3,2)(4,8,7,12,11,10)$
$(3,6,5,4)(7,8,9,10)$	$(1,4,2)(3,8,6)(5,10,7)(9,11,12)$
$(3,6,5,4)(7,8,9,10)$	$(1,6,3,2)(4,8)(5,9)(7,12,11,10)$
$(3,6,5,4)(7,8,9,10)$	$(1,7,3,2)(6,12,11,10)$
$(3,6,5,4)(7,8,9,10)$	$(1,9,6,3,2)(4,12,11,10,7)$
$(3,6,5,4)(7,8,9,10)$	$(1,5,9,6,3,2)(4,8,12,11,10,7)$

Table 5: Several pairs of generators of M_{12}

2.4 $(A_6, \mathrm{ASL}_3(2))$-group

Example 7.

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_3^{-1}, & a_1b_4^{-1}a_2^{-1}b_3, & a_2b_1a_2^{-1}b_2^{-1}, \\ a_2b_2a_3^{-1}b_1, & a_2b_3a_2^{-1}b_4, & a_2b_2^{-1}a_3b_1^{-1}, \\ a_3b_1a_3b_3^{-1}, & a_3b_2a_3b_4^{-1}, & a_3b_3a_3b_4 \end{array} \right\}.$$

Theorem 7. (1) *$P_h = A_6$, $P_v \cong \mathrm{ASL}_3(2) < S_8$.*

(2) *Any non-trivial normal subgroup of Γ has finite index.*

(3) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (2,4,3), \\ \rho_v(b_2) &= (3,5,4), \\ \rho_v(b_3) &= (1,2)(3,4), \\ \rho_v(b_4) &= (3,4)(5,6), \\ \\ \rho_h(a_1) &= (3,4)(5,6), \\ \rho_h(a_2) &= (1,7,8,2)(3,4,6,5), \\ \rho_h(a_3) &= (1,7,5,3)(2,8,6,4). \end{aligned}$$

(2) Note that

$$\mathrm{Stab}_{P_v}(\{1\}) = \langle (3,4)(5,6), (3,5,7)(4,6,8), (2,7,6,3)(4,8) \rangle \cong \mathrm{PSL}_3(2)$$

is a non-abelian simple group. The statement follows now from Proposition 9, or [15, Proposition 3.3.3] together with [16, Theorem 4.1], or directly from [16, Corollary 5.3].

(3) This is a short computation.

□

Conjecture 8. *Γ_0 is a simple group.*

3 Construction of (virtually) simple $(2m, 2n)$-groups

Non-residually finite $(2m, 2n)$-groups have been constructed in [14], [15], [16] for $2m = 196 = 14^2$, $2n = 324 = 18^2$ and independently in [69] for $2m = 8$, $2n = 6$ using completely different techniques. We first present in this section an irreducible (A_4, P_v)-group Γ with $P_v < S_{12}$ "quasi-primitive" but such that the quasi-center $\mathrm{QZ}(H_2)$ is not trivial. Applying a result of [16], this shows that Γ is non-residually finite (Example 8). Then, we embed Γ into a (A_6, A_{16})-group (Example 9), into a (A_8, A_{14})-group (Example 10), and into a $(\mathrm{ASL}_3(2)), A_{14})$-group (Example 11). All three examples turn out to be virtually simple (again by results of Burger and Mozes). Therefore, their minimal normal subgroup of finite index is a finitely presented torsion-free simple group. We think that this index is 4 in our examples. This is indeed true for a virtually simple (A_{10}, A_{10})-group (Example 14) constructed by means of an embedding of Daniel T. Wise's non-residually finite example (see [69] or Example 13). Therefore we get an explicit description of a finitely presented torsion-free simple group in $\mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$, which moreover has the form $F_9 *_{F_{81}} F_9$. Similar ideas lead to many other interesting groups in Section 3.7, 3.8 and 3.9.

3.1 A non-residually finite $(4, 12)$-group

We begin with a generalization of the notion of a primitive permutation group.

Definition. A non-trivial permutation group $G < \mathrm{Sym}(\Omega)$ of a set Ω is called *quasi-primitive*, if every non-trivial normal subgroup of G (in particular G itself) acts transitively on Ω.

Observe that primitive groups are quasi-primitive, which are for their part by definition transitive. Some structure theory for locally quasi-primitive groups of automorphisms of graphs has been developed in [15].

See the following list for all quasi-primitive, but not 2-transitive subgroups of S_{2n}, where $n \leq 8$:

G	$2n$	primitive	$\lvert G\rvert$	$G < A_{2n}$
A_5	10	Y	60	Y
S_5	10	Y	120	N
$\mathrm{PSL}_2(5)$	12	N	60	Y
$\mathrm{PSL}_2(7)$	14	N	168	Y
$2^4 : 5$	16	Y	80	Y
$2^4 : D_5$	16	Y	160	Y
$(A_4 \times A_4) : 2$	16	Y	288	Y
$(2^4 : 5) : 4$	16	Y	320	Y
$2^4 : 3^2 : 4$	16	Y	576	Y
$2^4 : S_3 \times S_3$	16	Y	576	Y
$2^4 : A_5$	16	Y	960	Y
$(S_4 \times S_4) : 2$	16	Y	1152	Y
$2^4 : S_5$	16	Y	1920	Y

Table 6: Quasi-primitive, not 2-transitive subgroups of S_{2n}, $n \leq 8$.

Example 8.

$$R(2,6) := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, \\ a_1b_3a_1^{-1}b_4^{-1}, & a_1b_4a_1^{-1}b_5^{-1}, \\ a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_2^{-1}, \\ a_1b_2^{-1}a_2b_3, & a_2b_1a_2^{-1}b_5^{-1}, \\ a_2b_2a_2b_3^{-1}, & a_2b_4a_2^{-1}b_4, \\ a_2b_5a_2^{-1}b_1^{-1}, & a_2b_6a_2^{-1}b_6 \end{array} \right\}.$$

Theorem 8. (1) *$P_h = A_4$, $P_v \cong \mathrm{PSL}_2(5) < S_{12}$, $|P_v| = 60$.*

(2) *Γ is irreducible.*

(3) *P_v is quasi-primitive, but not primitive.*

(4) *$\Lambda_2 \neq 1$, in particular $\mathrm{QZ}(H_2) \neq 1$.*

(5) *Γ is non-residually finite.*

(6) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(7) *(cf. [16, Theorem 6.4] where $m \geq 109$, $n \geq 175$) For every $m \geq 9$ and $n \geq 13$, there exists a torsion-free cocompact virtually simple lattice $\Lambda < U(A_{2m}) \times U(A_{2n})$ with dense projections.*

(8) *(cf. [16, Theorem 6.5]) Any $(2m, 2n)$-group injects for any even natural numbers $k \geq 4$, $\ell \geq 4$ in a virtually simple $(A_{4m+14+k}, A_{4n+22+\ell})$-group.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (), \\ \rho_v(b_2) &= (2,4,3), \\ \rho_v(b_3) &= (1,2,3), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (), \\ \rho_v(b_6) &= (), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,6,5,4,3)(7,8,9,10,11), \\ \rho_h(a_2) &= (1,5)(2,3)(4,9)(6,7)(8,12)(10,11). \end{aligned}$$

(2) Figure 6 shows that we can apply Proposition 1(3a) using the fact that $a_1b_1 = b_1a_1$ and that $\rho_v(b_3) = (1,2,3)$ acts transitively on $\{1,2,3\} \cong E_h \setminus \{a_1^{-1}\} = \{a_1, a_2, a_2^{-1}\}$. Note that the irreducibility criterion [16, Proposition 1.3] cannot be applied, since P_v is not primitive and K_h (defined in [16] or Section 4.9) is a 3-group.

(3) P_v is quasi-primitive, since it is simple and transitive. It has the non-trivial blocks $\{1,12\}$, $\{5,8\}$, $\{4,9\}$, $\{3,10\}$, $\{2,11\}$, $\{6,7\}$, and is therefore not primitive.

(4) $B := \{b_1^3, b_2^3, b_3^3, b_4^3, b_5^3, b_6^3, b_1^{-3}, b_2^{-3}, b_3^{-3}, b_4^{-3}, b_5^{-3}, b_6^{-3}\} \subset \Lambda_2$ by the subsequent Lemma 26(1b), since for each $b \in B$ and $a \in E_h$ we have $\rho_v(b)(a) = a$ and $\rho_h(a)(b) \in B$.

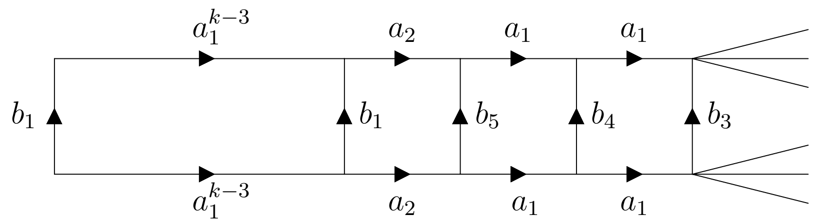


Figure 6: Proof of Theorem 8(2)

(5) We apply [16, Proposition 2.1].

(6) Easy computation.

(7) We imitate the proof of [16, Theorem 6.4], but replace ${}^{(0)}X = \mathcal{A}_{13,17}$ (see Example 40) by the (A_6, A_6)-complex of Example 1 and replace ${}^{(1)}X = \mathcal{A}_{13,17} \boxtimes \mathcal{A}_{13,17}$ by the non-residually finite $(4, 12)$-complex X. Note that we use in the proof that $\mathrm{PSL}_2(5) < S_{12}$ is even, i.e. a subgroup of A_{12}.

(8) If necessary, we embed the given $(2m, 2n)$-complex by [16, Proposition 6.2] in a $(4m, 4n)$-complex Y with *even* local permutation groups. Then we apply [16, Proposition 6.1] to the case where ${}^{(0)}X$ is the (A_6, A_6)-complex of Example 1, ${}^{(1)}X$ is the non-residually finite $(4, 12)$-complex X and ${}^{(2)}X = Y$.

□

Lemma 26. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R(m, n)\rangle$ *be a* $(2m, 2n)$*-group.*

(1a) *Let* $A \subset \langle a_1, \dots, a_m\rangle$. *If for each* $a \in A$ *and* $b \in E_v$ *we have* $\rho_h(a)(b) = b$ *and* $\rho_v(b)(a) \in A$, *then* $A \subset \Lambda_1$.

(1b) *Let* $B \subset \langle b_1, \dots, b_n\rangle$. *If for each* $b \in B$ *and* $a \in E_h$ *we have* $\rho_v(b)(a) = a$ *and* $\rho_h(a)(b) \in B$, *then* $B \subset \Lambda_2$.

Proof. The assumptions made in (1a) directly imply

$$A \subset \bigcap_{k \in \mathbb{N}} \ker \rho_h^{(k)}$$

and we are done since

$$\bigcap_{k \in \mathbb{N}} \ker \rho_h^{(k)} \cong \Lambda_1.$$

(1b) follows similarly. □

Conjecture 9.

$$\bigcap_{N \overset{f.i.}{\triangleleft} \Gamma} N = \Gamma_0.$$

See Table 7 for the orders of some quotients of Γ (the infinite quotients in this list correspond to elements in Λ_2).

A famous result proved by Malcev ([50], see also [47, Theorem IV.4.10]) says that every finitely generated residually finite group is Hopfian, where a group G is called *Hopfian*, if every epimorphism $G \to G$ is an isomorphism.

Question 3. *Is there a non-Hopfian* $(2m, 2n)$*-group?*

$\lvert\Gamma/\langle\langle w^k\rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w=a_1$	2	4	2	4	2	4	2	4	2	4	2	4
a_2	2	4	2	4	2	4	2	4	2	4	2	4
b_1	2	4	∞	4	2	∞	2	4	∞	4	2	∞
b_2	2	4	∞	4	2	∞	2	4	∞	4	2	∞
b_3	2	4	∞	4	2	∞	2	4	∞	4	2	∞
b_4	2	4	∞	4	2	∞	2	4	∞	4	2	∞
b_5	2	4	∞	4	2	∞	2	4	∞	4	2	∞
b_6	2	4	∞	4	2	∞	2	4	∞	4	2	∞

Table 7: Order of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$, $w \in \{a_1, a_2, b_1, \dots, b_6\}$, $k = 1, \dots, 12$, in Example 8

Remark. Obviously, $(2m, 2n)$-groups satisfying the normal subgroup theorem (Proposition 9) are Hopfian and by the result of Malcev, residually finite $(2m, 2n)$-groups (in particular reducible $(2m, 2n)$-groups) are also Hopfian. Note that Zlil Sela proved that torsion-free hyperbolic groups are Hopfian (see [65]).

3.2 A virtually simple (A_6, A_{16})-group

Example 9.

$$R(3,8) := \left\{ \begin{array}{lll} \underline{a_1b_1a_1^{-1}b_1^{-1}}, & \underline{a_1b_2a_2^{-1}b_3^{-1}}, & \underline{a_1b_3a_1^{-1}b_4^{-1}}, \\ \underline{a_1b_4a_1^{-1}b_5^{-1}}, & \underline{a_1b_5a_1^{-1}b_6^{-1}}, & \underline{a_1b_6a_1^{-1}b_2^{-1}}, \\ a_1b_7a_2b_8^{-1}, & a_1b_8a_2b_8, & a_1b_8^{-1}a_2b_7^{-1}, \\ a_1b_7^{-1}a_3^{-1}b_7, & \underline{a_1b_2^{-1}a_2b_3}, & \underline{a_2b_1a_2^{-1}b_5^{-1}}, \\ \underline{a_2b_2a_2b_3^{-1}}, & \underline{a_2b_4a_2^{-1}b_4}, & \underline{a_2b_5a_2^{-1}b_1^{-1}}, \\ \underline{a_2b_6a_2^{-1}b_6}, & a_2b_7a_3b_7^{-1}, & a_3b_1a_3^{-1}b_8, \\ a_3b_2a_3^{-1}b_2, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_1, \\ a_3b_5a_3^{-1}b_3, & a_3b_6a_3^{-1}b_6, & a_3b_8a_3^{-1}b_5 \end{array} \right\}.$$

Theorem 9. (1) $P_h = A_6$, $P_v = A_{16}$.

(2) Γ *is non-residually finite.*

(3) Γ *is a finitely presented torsion-free virtually simple group, in particular the minimal normal subgroup of finite index in* Γ

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N$$

is a finitely presented torsion-free simple group.

(4) *We have amalgam decompositions*

$$F_8 *_{F_{43}} F_{22} \cong \Gamma \cong F_3 *_{F_{33}} F_{17}$$

and

$$\mathrm{Aut}(\mathcal{T}_6) > F_{15} *_{F_{85}} F_{15} \cong \Gamma_0 \cong F_5 *_{F_{65}} F_5 < \mathrm{Aut}(\mathcal{T}_{16}).$$

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (), \\
\rho_v(b_2) &= (2,6,5), \\
\rho_v(b_3) &= (1,2,5), \\
\rho_v(b_4) &= (), \\
\rho_v(b_5) &= (), \\
\rho_v(b_6) &= (), \\
\rho_v(b_7) &= (1,5,3)(2,4,6), \\
\rho_v(b_8) &= (1,5)(2,6),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (2,6,5,4,3)(7,9,8)(11,12,13,14,15), \\
\rho_h(a_2) &= (1,5)(2,3)(4,13)(6,11)(8,10,9)(12,16)(14,15), \\
\rho_h(a_3) &= (1,13,14,5,9)(2,15)(3,12,8,16,4)(6,11).
\end{aligned}$$

(2) The embedding of the $(A_4, \mathrm{PSL}_2(5) < S_{12})$-complex of Example 8 into X (indicated by the twelve underlined geometric squares in $R(3,8)$) induces a π_1-injection by Proposition 5(1). Because of Theorem 8(5), Γ is non-residually finite neither.

(3) Apply [16, Corollary 5.4].

(4) We use the same arguments as in Section 2.

(5) Easy computations.

□

Conjecture 10. Γ_0 *is a finitely presented torsion-free simple group. Equivalently,*

$$\bigcap_{N \overset{f.i.}{\triangleleft} \Gamma} N = \Gamma_0.$$

3.3 A virtually simple (A_8, A_{14})-group

Example 10.

$$
R(4,7) := \left\{ \begin{array}{llll}
\underline{a_1b_1a_1^{-1}b_1^{-1}}, & \underline{a_1b_2a_2^{-1}b_3^{-1}}, & \underline{a_1b_3a_1^{-1}b_4^{-1}}, & \underline{a_1b_4a_1^{-1}b_5^{-1}}, \\
\underline{a_1b_5a_1^{-1}b_6^{-1}}, & \underline{a_1b_6a_1^{-1}b_2^{-1}}, & a_1b_7a_2^{-1}b_7^{-1}, & a_1b_7^{-1}a_3b_7, \\
\underline{a_1b_2^{-1}a_2b_3}, & \underline{a_2b_1a_2^{-1}b_5^{-1}}, & \underline{a_2b_2a_2b_3^{-1}}, & \underline{a_2b_4a_2^{-1}b_4}, \\
\underline{a_2b_5a_2^{-1}b_1^{-1}}, & \underline{a_2b_6a_2^{-1}b_6}, & a_2b_7a_4^{-1}b_7^{-1}, & a_3b_1a_4b_3^{-1}, \\
a_3b_2a_4b_1^{-1}, & a_3b_3a_4b_2, & a_3b_4a_3^{-1}b_5, & a_3b_5a_4b_4, \\
a_3b_6a_3^{-1}b_6^{-1}, & a_3b_7^{-1}a_4b_3, & a_3b_5^{-1}a_4^{-1}b_4^{-1}, & a_3b_3^{-1}a_4b_7, \\
a_3b_2^{-1}a_4b_2^{-1}, & a_3b_1^{-1}a_4b_1, & a_4b_6a_4^{-1}b_6^{-1}, & a_4b_5^{-1}a_4b_4^{-1}
\end{array} \right\}.
$$

Theorem 10. (1) *$P_h = A_8$, $P_v = A_{14}$.*

(2) *Γ is non-residually finite.*

(3) *Γ is a finitely presented torsion-free virtually simple group.*

(4) *There are amalgam decompositions*

$$F_7 *_{F_{49}} F_{25} \cong \Gamma \cong F_4 *_{F_{43}} F_{22}$$

and

$$\mathrm{Aut}(\mathcal{T}_8) > F_{13} *_{F_{97}} F_{13} \cong \Gamma_0 \cong F_7 *_{F_{85}} F_7 < \mathrm{Aut}(\mathcal{T}_{14}).$$

(5) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

Proof. (1)

$$
\begin{aligned}
\rho_v(b_1) &= (3,5)(4,6), \\
\rho_v(b_2) &= (2,8,7)(3,5)(4,6), \\
\rho_v(b_3) &= (1,2,7)(3,5)(4,6), \\
\rho_v(b_4) &= (3,4,5), \\
\rho_v(b_5) &= (4,5,6), \\
\rho_v(b_6) &= (), \\
\rho_v(b_7) &= (1,2,4,6)(3,8,7,5),
\end{aligned}
$$

$$
\begin{aligned}
\rho_h(a_1) &= (2,6,5,4,3)(9,10,11,12,13), \\
\rho_h(a_2) &= (1,5)(2,3)(4,11)(6,9)(10,14)(12,13), \\
\rho_h(a_3) &= (1,2,13,3)(4,10)(5,11)(8,12), \\
\rho_h(a_4) &= (2,13,14,12)(3,7)(4,10)(5,11).
\end{aligned}
$$

(2) The embedding of the $(A_4, \mathrm{PSL}_2(5) < S_{12})$-complex of Example 8 into X (indicated by the twelve underlined geometric squares in $R(4,7)$) induces a π_1-injection by Proposition 5(1).

(3) Apply [16, Corollary 5.4].

(4) Use the same arguments as in Section 2.

(5) These are easy computations.

$\square$

Conjecture 11. Γ_0 *is a finitely presented torsion-free simple group.*

Remark. It seems to be impossible to embed X of Example 8 into a virtually simple (A_6, A_{14})-complex. However, it seems to be easy to embed X of Example 8 into a virtually simple (A_{2m}, A_{2n})-complex, if $m \geq 3$, $n \geq 8$ or $m \geq 4$, $n \geq 7$.

3.4 A virtually simple $(\mathrm{ASL}_3(2)), A_{14})$-group

Example 11.

$$R(4,7) := \left\{ \begin{array}{llll} \underline{a_1 b_1 a_1^{-1} b_1^{-1}}, & \underline{a_1 b_2 a_2^{-1} b_3^{-1}}, & \underline{a_1 b_3 a_1^{-1} b_4^{-1}}, & \underline{a_1 b_4 a_1^{-1} b_5^{-1}}, \\ \underline{a_1 b_5 a_1^{-1} b_6^{-1}}, & \underline{a_1 b_6 a_1^{-1} b_2^{-1}}, & a_1 b_7 a_2^{-1} b_7^{-1}, & a_1 b_7^{-1} a_3 b_7, \\ \underline{a_1 b_2^{-1} a_2 b_3}, & \underline{a_2 b_1 a_2^{-1} b_5^{-1}}, & \underline{a_2 b_2 a_2 b_3^{-1}}, & \underline{a_2 b_4 a_2^{-1} b_4}, \\ \underline{a_2 b_5 a_2^{-1} b_1^{-1}}, & \underline{a_2 b_6 a_2^{-1} b_6}, & a_2 b_7 a_4^{-1} b_7^{-1}, & a_3 b_1 a_4 b_4, \\ a_3 b_2 a_3^{-1} b_3^{-1}, & a_3 b_3 a_4^{-1} b_2^{-1}, & a_3 b_4 a_4 b_7, & a_3 b_5 a_4 b_6^{-1}, \\ a_3 b_6 a_4 b_1^{-1}, & a_3 b_7^{-1} a_4 b_1, & a_3 b_6^{-1} a_4 b_5, & a_3 b_5^{-1} a_4 b_6, \\ a_3 b_4^{-1} a_4 b_5^{-1}, & a_3 b_3^{-1} a_4 b_2, & a_3 b_1^{-1} a_4 b_4^{-1}, & a_4 b_3 a_4 b_2^{-1} \end{array} \right\}.$$

Theorem 11. (1) $P_h \cong \mathrm{ASL}_3(2) < S_8$, $P_v = A_{14}$.

(2) Γ *is non-residually finite.*

(3) Γ *is a finitely presented torsion-free virtually simple group.*

(4) *There are amalgam decompositions*

$$F_7 *_{F_{49}} F_{25} \cong \Gamma \cong F_4 *_{F_{43}} F_{22}$$

and

$$\mathrm{Aut}(\mathcal{T}_8) > F_{13} *_{F_{97}} F_{13} \cong \Gamma_0 \cong F_7 *_{F_{85}} F_7 < \mathrm{Aut}(\mathcal{T}_{14}).$$

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (3,5)(4,6),\\
\rho_v(b_2) &= (2,8,7)(3,4,5),\\
\rho_v(b_3) &= (1,2,7)(4,6,5),\\
\rho_v(b_4) &= (3,5)(4,6),\\
\rho_v(b_5) &= (3,5)(4,6),\\
\rho_v(b_6) &= (3,5)(4,6),\\
\rho_v(b_7) &= (1,2,4,6)(3,8,7,5),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (2,6,5,4,3)(9,10,11,12,13),\\
\rho_h(a_2) &= (1,5)(2,3)(4,11)(6,9)(10,14)(12,13),\\
\rho_h(a_3) &= (1,6,5,11)(2,3)(4,14,8)(9,10)(12,13),\\
\rho_h(a_4) &= (1,11,7)(2,3)(4,10,9,14)(5,6)(12,13).
\end{aligned}$$

(2) The embedding of the $(A_4, \mathrm{PSL}_2(5) < S_{12})$-complex of Example 8 into X (indicated by the twelve underlined geometric squares in $R(4,7)$) induces a π_1-injection by Proposition 5(1).

(3) Apply [16, Corollary 5.3] (cf. Example 7).

(4) Use the same arguments as in Section 2.

(5) These are easy computations.

□

Conjecture 12. *Γ_0 is a finitely presented torsion-free simple group.*

3.5 Two examples of Daniel T. Wise

Example 12. *(See [69, Section II.2.1], the transition from his notations to ours is given by $x \to a_1$, $y \to a_2$, $a \to b_1$, $b \to b_2$, $c \to b_3$.)*

$$R(2,3) := \left\{ \begin{array}{ll} a_1 b_2 a_1^{-1} b_1^{-1}, & a_2 b_2 a_2^{-1} b_1^{-1}, \\ \\ a_1 b_3 a_2^{-1} b_3^{-1}, & a_1 b_1 a_2^{-1} b_2^{-1}, \\ \\ a_2 b_1 a_1^{-1} b_3^{-1}, & a_2 b_3 a_1^{-1} b_2^{-1} \end{array} \right\}.$$

Theorem 12. (1) *(Wise [69]) Γ is irreducible and not $\langle b_1, b_2, b_3 \rangle$-separable.*

(2) *$|\{b_3^{-n} a_2 a_1^{-1} b_3^n : n \in \mathbb{Z}\}| = \infty$, more precisely*

$$b_3^{-n} a_2 a_1^{-1} b_3^n = \begin{cases} a_1 a_2^{-1} b_2^{-n} b_3^n, & n \text{ odd} \\ a_2 a_1^{-1} b_2^{-n} b_3^n, & n \text{ even.} \end{cases}$$

(3)

$$b_3^{-n} (a_2 a_1^{-1})^2 b_3^n = \begin{cases} (a_1 a_2^{-1})^2, & n \text{ odd} \\ (a_2 a_1^{-1})^2, & n \text{ even.} \end{cases}$$

In particular,

$$|\{b_3^{-n} (a_2 a_1^{-1})^2 b_3^n : n \in \mathbb{Z}\}| = 2.$$

(4) $Z_\Gamma(b_3^2)$ *is not abelian, in particular* Γ *is not commutative transitive.*

(5) Γ *is not a CSA-group.*

Proof. (1) See [69]. Let G be a group and $H < G$ a subgroup. Recall that G is said to be H*-separable*, if for each element $g \in G \setminus H$, there is a homomorphism $\psi : G \to Q$ onto a finite group Q such that $\psi(g) \notin \psi(H)$. It is shown in [69, Corollary II.4.4] that $\psi(a_1 a_2^{-1}) \in \psi(\langle b_1, b_2, b_3 \rangle)$ for every homomorphism $\psi : \Gamma \to Q$ with $|Q| < \infty$.

(2) This can be proved by induction, using $b_3 a_1 a_2^{-1} = a_2 a_1^{-1} b_2$ and $b_3 a_2 a_1^{-1} = a_1 a_2^{-1} b_2$.

(3) This follows from the fact that $(a_2 a_1^{-1})^2$ and b_3^2 commute.

(4) This follows from $\{b_3, (a_2 a_1^{-1})^2\} \subset Z_\Gamma(b_3^2)$ and $[b_3, (a_2 a_1^{-1})^2] = (a_1 a_2^{-1})^4 \neq 1$. A group G is called *commutative transitive*, if $[g_1, g_2] = [g_2, g_3] = 1$ $(g_1, g_2, g_3 \neq 1)$ always implies $[g_1, g_3] = 1$.

(5) A group G is called a *CSA-group* if every maximal abelian subgroup M of G is malnormal, i.e. $g^{-1} M g \cap M = 1$ for any $g \in G \setminus M$. Here, $\langle b_3 \rangle$ is a maximal abelian subgroup of Γ, since $\langle b_3 \rangle = Z_\Gamma(b_3)$ by Proposition 8(1b), but $(a_2 a_1^{-1})^{-2} b_3^2 (a_2 a_1^{-1})^2 = b_3^2 \in \langle b_3 \rangle$. It is known (see [58]) that all torsion-free hyperbolic groups are CSA-groups and that all CSA-groups are commutative transitive (in particular (5) is directly implied by (4)).

□

Remark. The proof of Theorem 12(1) given in [69] is based on the fact that the elements a_2, b_3 have no commuting non-trivial powers (this phenomenon is called *anti-torus* and was proved in [69, Proposition II.3.8]. More about anti-tori in Section 5.6). Note however, that $\langle a_2, b_3 \rangle$ is not a free subgroup of Γ since we have a non-trivial relation $b_3^{-2} a_2^{-3} b_3^2 a_2 b_3^{-1} a_2 b_3 a_2 = 1$ in Γ.

Remark. $\{(a_1 a_2^{-1})^2, (a_2^{-1} a_1)^2\} \subset \Lambda_1$.

Using the separability property of Theorem 12(1) and the following lemma of Darren D. Long and Graham A. Niblo, a doubling of the $(4,6)$-group Γ along its subgroup $\langle b_1, b_2, b_3 \rangle$ (geometrically: doubling X along its vertical 1-skeleton E_v) leads to the non-residually finite $(8,6)$-group of Example 13. (By a *double* of a group G along a subgroup H, we mean an amalgamated free product $G *_{H=\bar{H}} \bar{G}$, where $\bar{G} \hookleftarrow \bar{H}$ is an isomorphic copy of $G \hookleftarrow H$.)

Lemma 27. *(see [42, Lemma, p.211]) Let* $\theta : G \to G$ *be an automorphism of a residually finite group* G*. Then* G *is* $\mathrm{Fix}(\theta)$*-separable, where* $\mathrm{Fix}(\theta) := \{g \in G : \theta(g) = g\}$*. More precisely, if* $\theta : G \to G$ *is an automorphism and* G *is not* $\mathrm{Fix}(\theta)$*-separable, then*

$$x^{-1}\theta(x) \in \bigcap_{N \overset{f.i.}{\lhd} G} N,$$

where $x \in G \setminus \mathrm{Fix}(\theta)$ *is any element such that* $\psi(x) \in \psi(\mathrm{Fix}(\theta))$ *for all homomorphisms* $\psi : G \to Q$ *onto finite groups* Q*.*

Proof. See [42]. The same result is true for endomorphisms $\theta : G \to G$ of finitely generated residually finite groups G, see [69, Theorem II.5.2]. □

Example 13. *(See [69, Section II.5], where this example is called D)*

$$R(4,3) := \left\{ \begin{array}{cccc} \underline{a_1 b_2 a_1^{-1} b_1^{-1}}, & \underline{a_2 b_2 a_2^{-1} b_1^{-1}}, & \underline{a_1 b_3 a_2^{-1} b_3^{-1}}, & \underline{a_1 b_1 a_2^{-1} b_2^{-1}}, \\ \underline{a_2 b_1 a_1^{-1} b_3^{-1}}, & \underline{a_2 b_3 a_1^{-1} b_2^{-1}}, & a_3 b_2 a_3^{-1} b_1^{-1}, & a_4 b_2 a_4^{-1} b_1^{-1}, \\ a_3 b_3 a_4^{-1} b_3^{-1}, & a_3 b_1 a_4^{-1} b_2^{-1}, & a_4 b_1 a_3^{-1} b_3^{-1}, & a_4 b_3 a_3^{-1} b_2^{-1} \end{array} \right\}.$$

Theorem 13. *(Wise [69, Main Theorem II.5.5])* Γ *is non-residually finite.*

Proof. By [69], we have for example

$$a_2 a_1^{-1} a_3 a_4^{-1} \in \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

$\square$

Remark. For each $k \in \mathbb{N}$, we can find a normal subgroup of Γ of index k. Indeed, it is easy to check that

$$\Gamma / \langle\langle b_1, b_2, b_3, a_1, a_2, a_3^k \rangle\rangle_\Gamma = \langle a_3 \mid a_3^k \rangle \cong \mathbb{Z}_k.$$

In particular, it follows that

$$[\Gamma : \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N] = \infty.$$

Of course (by Theorem 13): $a_2 a_1^{-1} a_3 a_4^{-1} \in \langle\langle b_1, b_2, b_3, a_1, a_2, a_3^k \rangle\rangle_\Gamma$. Indeed, we even have $a_2 a_1^{-1} a_3 a_4^{-1} \in \langle\langle b_3 \rangle\rangle_\Gamma$, since

$$a_2 a_1^{-1} a_3 a_4^{-1} = (b_3^{-1} a_1 b_3 a_1^{-1} b_3)(a_4 b_3^{-1} a_4^{-1}).$$

Remark. $\Gamma / \langle\langle b_1 \rangle\rangle_\Gamma = \langle a_1, a_3 \rangle \cong F_2$, in particular, Γ has "many" quotients.

Remark. In Example 12 and 13, the permutation groups P_h and P_v are not transitive and Γ^{ab} are infinite ($\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ respectively), due to the fact that the groups Γ are horizontally and vertically directed.

3.6 Construction of a simple amalgam $F_9 *_{F_{81}} F_9$

In this section, we construct a finitely presented torsion-free simple group. First, we give two elementary but crucial lemmas used in the proof of Theorem 14(6).

Lemma 28. *Let G be a group, $H < G$ a non-residually finite subgroup of G and $h \in H$ an element such that*

$$1 \neq h \in \bigcap_{M \overset{f.i.}{\lhd} H} M.$$

Then

$$h \in \bigcap_{N \overset{f.i.}{\lhd} G} N$$

(in particular G is again non-residually finite).

Proof. Let $N \lhd G$ be a normal subgroup of finite index in G. Obviously, $N \cap H \lhd G \cap H = H$. Moreover $[H : (N \cap H)] \leq [G : N] < \infty$ by Lemma 12, hence $h \in N \cap H < N$. $\square$

Lemma 29. *Let G be a non-residually finite group and $g \in G$ an element such that*

$$1 \neq g \in \bigcap_{N \overset{f.i.}{\lhd} G} N$$

and assume that the normal subgroup $\langle\langle g \rangle\rangle_G$ has finite index in G. Then

$$\langle\langle g \rangle\rangle_G = \bigcap_{N \overset{f.i.}{\lhd} G} N.$$

Proof. By assumption, we have $\langle\langle g \rangle\rangle_G \overset{\text{f.i.}}{\lhd} G$, hence

$$\langle\langle g \rangle\rangle_G \supseteq \bigcap_{N \overset{\text{f.i.}}{\lhd} G} N.$$

The other inclusion follows directly from

$$g \in \bigcap_{N \overset{\text{f.i.}}{\lhd} G} N \ \lhd G.$$

$\square$

Now, we are ready to describe one of our main examples:

Example 14.

$$R(5,5) := \left\{ \begin{array}{lllll} \underline{\underline{a_1 b_1 a_2^{-1} b_2^{-1}}}, & \underline{\underline{a_1 b_2 a_1^{-1} b_1^{-1}}}, & \underline{\underline{a_1 b_3 a_2^{-1} b_3^{-1}}}, & a_1 b_4 a_2 b_5^{-1}, & a_1 b_5 a_5^{-1} b_4, \\ a_1 b_5^{-1} a_3 b_4^{-1}, & a_1 b_4^{-1} a_3 b_5, & \underline{\underline{a_1 b_3^{-1} a_2^{-1} b_2}}, & \underline{\underline{a_1 b_1^{-1} a_2^{-1} b_3}}, & \underline{\underline{a_2 b_2 a_2^{-1} b_1^{-1}}}, \\ a_2 b_4 a_2^{-1} b_5, & a_2 b_5 a_4 b_4^{-1}, & \underline{a_3 b_1 a_4^{-1} b_2^{-1}}, & \underline{a_3 b_2 a_3^{-1} b_1^{-1}}, & \underline{a_3 b_3 a_4^{-1} b_3^{-1}}, \\ a_3 b_4 a_4 b_5, & a_3 b_5^{-1} a_4 b_4, & \underline{a_3 b_3^{-1} a_4^{-1} b_2}, & \underline{a_3 b_1^{-1} a_4^{-1} b_3}, & \underline{a_4 b_2 a_4^{-1} b_1^{-1}}, \\ a_4 b_5^{-1} a_5^{-1} b_4^{-1}, & a_5 b_1 a_5^{-1} b_3, & a_5 b_2 a_5^{-1} b_5^{-1}, & a_5 b_3 a_5^{-1} b_1^{-1}, & a_5 b_4 a_5^{-1} b_2^{-1}, \end{array} \right\}.$$

Theorem 14. (1) *$P_h = A_{10}$, $P_v = A_{10}$.*

(2) *Γ is non-residually finite.*

(3) *Γ is a finitely presented torsion-free virtually simple group.*

(4) *There are two amalgam decompositions*

$$\Gamma \cong F_5 *_{F_{41}} F_{21}$$

and two amalgam decompositions

$$\Gamma_0 \cong F_9 *_{F_{81}} F_9 < \mathrm{Aut}(\mathcal{T}_{10}).$$

(5) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(6) *Γ_0 is a finitely presented torsion-free simple group!*

(7) *$d(\Gamma^k) \to \infty$ linearly for $k \to \infty$, but $d(\Gamma_0^k) \leq 3$ for all $k \in \mathbb{N}$.*

(8) *$Z_\Gamma(a_5) = N_\Gamma(\langle a_5 \rangle) = \langle a_5 \rangle$.*

(9) *$b_1 \in Z_\Gamma(a_5^4)$, in particular Γ is not commutative transitive.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (7,8)(9,10),\\
\rho_v(b_2) &= (1,2)(3,4),\\
\rho_v(b_3) &= (1,2)(3,4)(7,8)(9,10),\\
\rho_v(b_4) &= (1,8,4,5)(2,7,3,10),\\
\rho_v(b_5) &= (1,9,4,8)(3,10,6,7),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,2)(4,6,7,5)(8,10,9),\\
\rho_h(a_2) &= (1,2,3)(4,5,7,6)(9,10),\\
\rho_h(a_3) &= (1,2)(4,5,7,6)(8,10,9),\\
\rho_h(a_4) &= (1,2,3)(4,6,7,5)(9,10),\\
\rho_h(a_5) &= (1,3,10,8)(2,4,6,9,7,5).
\end{aligned}$$

(2) The embedding of the non-residually finite complex of Example 13 into X, indicated by the twelve (single or double) underlined geometric squares in $R(5,5)$, induces a π_1-injection by Proposition 5(1). The six geometric squares coming from Example 12 are doubly underlined.

(3) Apply [16, Corollary 5.4].

(4) We use the same methods as usual.

(5) These are easy computations.

(6) We only have to show that

$$\Gamma_0 = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

Set $w := a_2 a_1^{-1} a_3 a_4^{-1}$. Then by Theorem 13 and Lemma 28 we have

$$w \in \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N,$$

hence by Lemma 29, using the fact that every non-trivial normal subgroup of Γ has finite index in Γ (by Proposition 9), we have

$$\langle\langle w \rangle\rangle_\Gamma = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

A computer algebra system like GAP ([28]) immediately checks that

$$[\Gamma : \langle\langle w \rangle\rangle_\Gamma] = |\langle a_1, \dots, a_5, b_1, \dots, b_5 \mid R(5,5), w \rangle| = 4.$$

Since $[\Gamma : \Gamma_0] = 4$ and $w \in \Gamma_0$, we conclude that

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N = \langle\langle w \rangle\rangle_\Gamma = \Gamma_0.$$

(Alternatively and more explicitly, one proves $\langle\langle w \rangle\rangle_\Gamma = \Gamma_0$ by checking that

$$\Gamma_0 = \langle a_1 a_2^{-1}, b_3 b_1^{-1}, b_3 b_5^{-1} \rangle$$

and

$$
\begin{aligned}
a_1a_2^{-1} &= (b_2b_5wb_5^{-1}b_2^{-1})(b_5w^{-1}b_5^{-1}) \in \langle\!\langle w \rangle\!\rangle_\Gamma, \\
b_3b_1^{-1} &= (b_1^{-1}b_5w^{-1}b_5^{-1}b_1)(b_1b_5wb_5^{-1}b_1^{-1}) \in \langle\!\langle w \rangle\!\rangle_\Gamma, \\
b_3b_5^{-1} &= (b_1^{-1}b_4^{-1}wb_4b_1)(b_5b_4^{-1}w^{-1}b_4b_5^{-1}) \in \langle\!\langle w \rangle\!\rangle_\Gamma.)
\end{aligned}
$$

(7) We apply results of James Wiegold and John S. Wilson ([68]). First note that $d(\Gamma) = 2$, since for example $\Gamma = \langle a_1, b_4 \rangle$, and $d(\Gamma_0) = 2$, since $\Gamma_0 = \langle a_1^2, b_5b_1^{-1} \rangle$ (this can be checked with GAP). By [68, Theorem 2.2], we have $d(\Gamma^k) = 2k$, if $k \geq 18$. However, using the simplicity of Γ_0, [68, Theorem 4.3] implies $d(\Gamma_0^k) \leq d(\Gamma_0) + 1 = 3$ for all $k \in \mathbb{N}$.

(8) This follows from Proposition 8.

(9) We compute $a_5^4b_1 = b_1a_5^4$. Combining this with (8), Γ is not commutative transitive.

□

A finite presentation of the simple group Γ_0 is given as follows: We take 37 generators $s_1, \ldots, s_{37}$ and 100 relators

$$
\left\{
\begin{array}{lllll}
s_{24}s_{34}, & s_{10}s_{23}s_{33}, & s_{11}s_{24}s_{35}, & s_{12}s_{19}s_{37}, & s_{13}s_{27}s_{31}, \\
s_{18}s_{20}s_{36}, & s_{17}s_{20}s_{32}, & s_{16}s_{24}s_{29}, & s_{14}s_{24}s_{30}, & s_1s_{10}s_{24}s_{33}, \\
s_1s_{12}s_{24}s_{32}, & s_1s_{13}s_{21}s_{36}, & s_2s_{26}s_{34}, & s_2s_{10}s_{25}s_{33}, & s_2s_{11}s_{26}s_{35}, \\
s_2s_{12}s_{21}s_{32}, & s_2s_{18}s_{21}s_{31}, & s_2s_{16}s_{26}s_{29}, & s_2s_{14}s_{26}s_{30}, & s_3s_{10}s_{26}s_{33}, \\
s_3s_{18}s_{27}s_{36}, & s_4s_{27}s_{30}, & s_4s_{10}s_{27}s_{37}, & s_4s_{11}s_{27}s_{33}, & s_4s_{12}s_{27}s_{34}, \\
s_5s_{10}s_{19}s_{33}, & s_5s_{34}, & s_5s_{11}s_{19}s_{35}, & s_5s_{13}s_{24}s_{36}, & s_5s_{17}s_{22}s_{37}, \\
s_5s_{12}s_{25}s_{32}, & s_5s_{18}s_{25}s_{31}, & s_5s_{15}s_{19}s_{30}, & s_5s_{16}s_{19}s_{28}, & s_6s_{19}s_{34}, \\
s_6s_{18}s_{19}s_{36}, & s_6s_{12}s_{26}s_{37}, & s_7s_{10}s_{21}s_{33}, & s_7s_{20}s_{34}, & s_7s_{11}s_{21}s_{35}, \\
s_7s_{18}s_{26}s_{36}, & s_7s_{17}s_{26}s_{32}, & s_7s_{15}s_{21}s_{30}, & s_7s_{16}s_{21}s_{28}, & s_8s_{21}s_{34}, \\
s_8s_{12}s_{22}s_{32}, & s_9s_{16}s_{22}s_{33}, & s_9s_{13}s_{22}s_{34}, & s_9s_{22}s_{35}, & s_9s_{10}s_{22}s_{36}, \\
s_6s_{15}s_{28}, & s_5s_{14}s_{29}, & s_6s_{16}s_{30}, & s_1s_{18}s_{31}, & s_9s_{12}s_{32}, \\
s_2s_{17}s_{37}, & s_2s_{13}s_{36}, & s_6s_{10}s_{35}, & s_6s_{11}s_{33}, & s_6s_{14}s_{19}s_{29}, \\
s_6s_{13}s_{19}s_{31}, & s_3s_{17}s_{19}s_{32}, & s_8s_{15}s_{20}s_{28}, & s_7s_{14}s_{20}s_{29}, & s_8s_{16}s_{20}s_{30}, \\
s_3s_{13}s_{20}s_{31}, & s_3s_{12}s_{20}s_{37}, & s_8s_{10}s_{20}s_{35}, & s_8s_{11}s_{20}s_{33}, & s_8s_{14}s_{21}s_{29}, \\
s_9s_{17}s_{21}s_{37}, & s_9s_{11}s_{22}s_{28}, & s_9s_{18}s_{22}s_{29}, & s_9s_{14}s_{22}s_{30}, & s_9s_{15}s_{22}s_{31}, \\
s_1s_{14}s_{23}s_{29}, & s_{15}s_{23}s_{28}, & s_1s_{16}s_{23}s_{30}, & s_6s_{17}s_{23}s_{32}, & s_4s_{18}s_{23}s_{36}, \\
s_7s_{13}s_{23}s_{31}, & s_7s_{12}s_{23}s_{37}, & s_1s_{11}s_{23}s_{34}, & s_1s_{23}s_{35}, & s_1s_{15}s_{24}s_{28}, \\
s_1s_{17}s_{24}s_{37}, & s_8s_{18}s_{24}s_{31}, & s_3s_{14}s_{25}s_{29}, & s_2s_{15}s_{25}s_{28}, & s_3s_{16}s_{25}s_{30}, \\
s_8s_{17}s_{25}s_{37}, & s_8s_{13}s_{25}s_{36}, & s_3s_{11}s_{25}s_{34}, & s_3s_{25}s_{35}, & s_3s_{15}s_{26}s_{28}, \\
s_4s_{13}s_{26}s_{31}, & s_4s_{14}s_{27}s_{35}, & s_4s_{15}s_{27}s_{32}, & s_4s_{16}s_{27}s_{28}, & s_4s_{17}s_{27}s_{29}
\end{array}
\right\}.
$$

Of course, this presentation can be slightly simplified, e.g. using the identities $s_5 = s_{24} = s_{34}^{-1}$.

Remark. The smallest candidate for being a finitely presented torsion-free simple group in the construction leading to [16, Theorem 6.4] has much more complicated amalgam decompositions

$$\mathrm{Aut}(\mathcal{T}_{218}) > F_{349} *_{F_{75865}} F_{349} \cong F_{217} *_{F_{75601}} F_{217} < \mathrm{Aut}(\mathcal{T}_{350}).$$

3.7 More simple groups

Using exactly the same ideas as in Theorem 14, we embed in this section the non-residually finite $(8, 6)$-complex of Example 13 into several $(2m, 2n)$-complexes with virtually simple fundamental groups Γ. See the following list (Table 8) for $(2m, 2n) \in \{(10,10), (10,12), (12,8), (12,10), (12,12)\}$. As before, the group

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N = \langle\!\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\!\rangle_\Gamma$$

is finitely presented torsion-free and simple. In the list, we use the following notation:

$$\Gamma^* := \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

In the third column, $[2,2]$ stands for $\mathbb{Z}_2^2$ etc. and in the last column, for example $(9, 81, 9)$ means an amalgam decomposition $F_9 *_{F_{81}} F_9$. Note that Γ_0 and Γ^* always have two amalgam decompositions, a horizontal and a vertical one. Since $\Gamma^* < \Gamma_0$ is a subgroup, the index $[\Gamma : \Gamma^*]$ is a multiple of 4. In most (but not all) examples listed below, we have $[\Gamma, \Gamma] = \Gamma^*$, in particular for these examples $|\Gamma^{ab}| = [\Gamma : \Gamma^*]$ and $[\Gamma, \Gamma]$ is simple. In all examples (in particular for those with $\Gamma^* \subsetneqq [\Gamma, \Gamma]$), we compute

$$\Gamma^* = \langle\!\langle [a_1, a_2], [a_1, b_1], [a_1, b_2], [a_1, b_3], [a_2, b_1], [a_2, b_2], [a_2, b_3], [b_1, b_2], [b_1, b_3], [b_2, b_3] \rangle\!\rangle_\Gamma.$$

This observation gives some indications that Conjecture 16 (in Section 3.10) is true. If $[\Gamma : \Gamma^*] > |\Gamma^{ab}|$, we give the non-abelian quotient Γ/Γ^*, which is not always nilpotent. Three more examples of the list (Example 15, Example 16 and Example 17) will be described explicitly after the list. We have chosen these examples for the following reasons: In Example 15, $P_h \cong M_{12}$, the fascinating Mathieu group; in Example 16, $\Gamma^* \subsetneqq [\Gamma, \Gamma]$ and in Example 17, $[\Gamma : \Gamma^*] = 40$ is relatively large, the largest in this list.

Ex	Γ	Γ^{ab}	$\lvert\Gamma^{ab}\rvert$	$[\Gamma : \Gamma^*]$	Γ/Γ^*	$\Gamma^* = \langle\!\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\!\rangle_\Gamma$
	$(10,10)$					
14	(A_{10}, A_{10})	$[2,2]$	4	4		$(9,81,9) = (9,81,9)$
	(A_{10}, A_{10})	$[2,2,2]$	8	8		$(17,161,17) = (17,161,17)$
	(A_{10}, A_{10})	$[2,4]$	8	8		$(17,161,17) = (17,161,17)$
	(A_{10}, A_{10})	$[2,2,3]$	12	12		$(25,241,25) = (25,241,25)$
	(A_{10}, A_{10})	$[2,2,4]$	16	16		$(33,321,33) = (33,321,33)$
	(A_{10}, A_{10})	$[2,8]$	16	16		$(33,321,33) = (33,321,33)$
	(A_{10}, A_{10})	$[2,2,5]$	20	20		$(41,401,41) = (41,401,41)$
	(A_{10}, A_{10})	$[2,2,2,3]$	24	24		$(49,481,49) = (49,481,49)$
	(A_{10}, A_{10})	$[2,3,4]$	24	24		$(49,481,49) = (49,481,49)$
	(A_{10}, A_{10})	$[2,2,8]$	32	32		$(65,641,65) = (65,641,65)$
17	(A_{10}, A_{10})	$[2,4,5]$	40	40		$(81,801,81) = (81,801,81)$
	$(10,12)$					
	(A_{10}, A_{12})	$[2,2]$	4	4		$(11,101,11) = (9,97,9)$
16	(A_{10}, A_{12})	$[2,2]$	4	12	D_6	$(31,301,31) = (25,289,25)$
	(A_{10}, A_{12})	$[2,2,2]$	8	8		$(21,201,21) = (17,193,17)$
	(A_{10}, A_{12})	$[2,2,2]$	8	24	$S_3 \times \mathbb{Z}_2^2$	$(61,601,61) = (49,577,49)$
	(A_{10}, A_{12})	$[2,4]$	8	8		$(21,201,21) = (17,193,17)$
	$(12,8)$					
	(A_{12}, A_8)	$[2,2]$	4	4		$(7,73,7) = (11,81,11)$
	(A_{12}, A_8)	$[2,4]$	8	8		$(13,145,13) = (21,161,21)$
15	(M_{12}, A_8)	$[2,2]$	4	4		$(7,73,7) = (11,81,11)$

	$(12,10)$					
	(A_{12}, A_{10})	$[2,2]$	4	4		$(9,97,9) = (11,101,11)$
	(A_{12}, A_{10})	$[2,2]$	4	12	D_6	$(25,289,25) = (31,301,31)$
	(A_{12}, A_{10})	$[2,2]$	4	20	$D_5 \times \mathbb{Z}_2$	$(41,481,41) = (51,501,51)$
	(A_{12}, A_{10})	$[2,2,2]$	8	8		$(17,193,17) = (21,201,21)$
	(A_{12}, A_{10})	$[2,4]$	8	8		$(17,193,17) = (21,201,21)$
	(A_{12}, A_{10})	$[2,2,2]$	8	16	$D_4 \times \mathbb{Z}_2$	$(33,385,33) = (41,401,41)$
	(A_{12}, A_{10})	$[2,2,3]$	12	12		$(25,289,25) = (31,301,31)$
	(A_{12}, A_{10})	$[2,8]$	16	16		$(33,385,33) = (41,401,41)$
	(A_{12}, A_{10})	$[2,2,5]$	20	20		$(41,481,41) = (51,501,51)$
	(A_{12}, A_{10})	$[2,2,2,3]$	24	24		$(49,577,49) = (61,601,61)$
	(M_{12}, A_{10})	$[2,2]$	4	4		$(9,97,9) = (11,101,11)$
	$(12,12)$					
	(A_{12}, A_{12})	$[2,2]$	4	4		$(11,121,11) = (11,121,11)$
	(A_{12}, A_{12})	$[2,2,2]$	8	8		$(21,241,21) = (21,241,21)$
	(A_{12}, A_{12})	$[2,2,3]$	12	12		$(31,361,31) = (31,361,31)$

Table 8: Many simple groups Γ^*

Example 15.

$$R(6,4) := \left\{ \begin{array}{llllll} \underline{\underline{a_1 b_1 a_2^{-1} b_2^{-1}}}, & \underline{\underline{a_1 b_2 a_1^{-1} b_1^{-1}}}, & \underline{\underline{a_1 b_3 a_2^{-1} b_3^{-1}}}, & a_1 b_4 a_3 b_4, & a_1 b_4^{-1} a_2 b_4^{-1}, & \underline{\underline{a_1 b_3^{-1} a_2^{-1} b_2}}, \\ \underline{\underline{a_1 b_1^{-1} a_2^{-1} b_3}}, & \underline{\underline{a_2 b_2 a_2^{-1} b_1^{-1}}}, & a_2 b_4 a_5 b_4, & \underline{a_3 b_1 a_4^{-1} b_2^{-1}}, & \underline{a_3 b_2 a_3^{-1} b_1^{-1}}, & \underline{a_3 b_3 a_4^{-1} b_3^{-1}}, \\ a_3 b_4^{-1} a_4^{-1} b_4^{-1}, & \underline{a_3 b_3^{-1} a_4^{-1} b_2}, & \underline{a_3 b_1^{-1} a_4^{-1} b_3}, & \underline{a_4 b_2 a_4^{-1} b_1^{-1}}, & a_4 b_4^{-1} a_5 b_4^{-1}, & a_5 b_1 a_6^{-1} b_2, \\ a_5 b_2 a_6^{-1} b_2^{-1}, & a_5 b_3 a_5^{-1} b_3^{-1}, & a_5 b_2^{-1} a_6^{-1} b_1^{-1}, & a_5 b_1^{-1} a_6^{-1} b_1, & a_6 b_3 a_6^{-1} b_4^{-1}, & a_6 b_4 a_6^{-1} b_3 \end{array} \right\}.$$

Theorem 15. (1) *$P_h \cong M_{12}$, $P_v = A_8$.*

(2) *Γ is non-residually finite.*

(3) *Γ is a finitely presented torsion-free virtually simple group.*

(4) *There are amalgam decompositions*

$$F_4 *_{F_{37}} F_{19} \cong \Gamma \cong F_6 *_{F_{41}} F_{21}$$

and

$$\mathrm{Aut}(\mathcal{T}_{12}) > F_7 *_{F_{73}} F_7 \cong \Gamma_0 \cong F_{11} *_{F_{81}} F_{11} < \mathrm{Aut}(\mathcal{T}_8).$$

(5) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(6) *Γ_0 is a finitely presented torsion-free simple group.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (5,6)(7,8)(9,10)(11,12),\\
\rho_v(b_2) &= (1,2)(3,4)(5,6)(7,8),\\
\rho_v(b_3) &= (1,2)(3,4)(9,10)(11,12),\\
\rho_v(b_4) &= (1,11,5,9,10)(2,12,3,4,8),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,2)(4,5)(6,8,7),\\
\rho_h(a_2) &= (1,2,3)(4,5)(7,8),\\
\rho_h(a_3) &= (1,2)(4,5)(6,8,7),\\
\rho_h(a_4) &= (1,2,3)(4,5)(7,8),\\
\rho_h(a_5) &= (1,7)(4,5),\\
\rho_h(a_6) &= (2,8)(3,5,6,4).
\end{aligned}$$

(2) The embedding of the non-residually finite complex of Example 13 into X (indicated by the twelve underlined geometric squares in $R(6,4)$) induces a π_1-injection by Proposition 5(1).

(3) We use [16, Corollary 5.3] and conclude as in [16, Corollary 5.4].

(4) Same arguments as usual.

(5) These are easy computations.

(6) The proof is in the same spirit as the proof of Theorem 14(6).

$\square$

Example 16.

$$R(5,6) := \left\{ \begin{array}{lllll}
\underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_2b_4^{-1}, & a_1b_5a_2b_5^{-1}, \\
a_1b_6a_4^{-1}b_4, & a_1b_6^{-1}a_4b_6, & a_1b_5^{-1}a_2^{-1}b_5, & a_1b_4^{-1}a_4^{-1}b_6^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}}, \\
\underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, & a_2b_4a_3^{-1}b_6^{-1}, & a_2b_6a_3^{-1}b_4^{-1}, & a_2b_6^{-1}a_3b_6, \\
\underline{a_3b_1a_4^{-1}b_2^{-1}}, & \underline{a_3b_2a_3^{-1}b_1^{-1}}, & \underline{a_3b_3a_4^{-1}b_3^{-1}}, & a_3b_4a_5b_5, & a_3b_5a_4^{-1}b_4^{-1}, \\
a_3b_5^{-1}a_4^{-1}b_5^{-1}, & \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, & \underline{a_4b_2a_4^{-1}b_1^{-1}}, & a_4b_4^{-1}a_5b_5^{-1}, \\
a_5b_1a_5^{-1}b_1^{-1}, & a_5b_2a_5^{-1}b_2, & a_5b_3a_5^{-1}b_5, & a_5b_4a_5^{-1}b_3^{-1}, & a_5b_6a_5^{-1}b_6
\end{array} \right\}.$$

Theorem 16. *Let*

$$\Gamma^* := \bigcap_{N \overset{f.i.}{\triangleleft} \Gamma} N.$$

(1) $P_h = A_{10}$, $P_v = A_{12}$.

(2) Γ^* *is a finitely presented torsion-free simple group.*

(3) *The subgroups of* Γ *of finite index and the normal subgroups of* Γ *are completely known.*

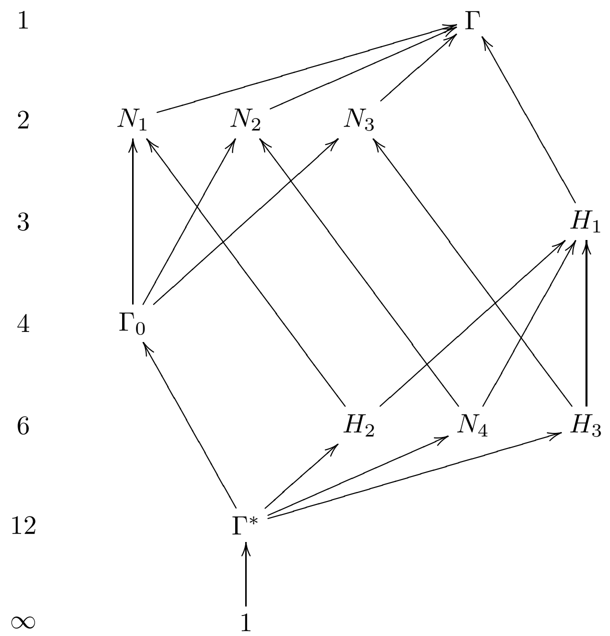
Proof. (1)

$$
\begin{aligned}
\rho_v(b_1) &= (7,8)(9,10),\\
\rho_v(b_2) &= (1,2)(3,4),\\
\rho_v(b_3) &= (1,2)(3,4)(7,8)(9,10),\\
\rho_v(b_4) &= (1,9,8,5,7,10,2,3,4),\\
\rho_v(b_5) &= (1,9,10,2)(3,4,6)(7,8),\\
\rho_v(b_6) &= (1,4,10,7)(2,3,9,8),
\end{aligned}
$$

$$
\begin{aligned}
\rho_h(a_1) &= (1,2)(6,9)(10,12,11),\\
\rho_h(a_2) &= (1,2,3)(4,6)(11,12),\\
\rho_h(a_3) &= (1,2)(4,5,8)(7,9)(10,12,11),\\
\rho_h(a_4) &= (1,2,3)(4,7)(5,9,8)(11,12),\\
\rho_h(a_5) &= (2,11)(3,4,8)(5,10,9)(6,7).
\end{aligned}
$$

(2) The proof is the same as in the previous two theorems.

(3) We have used GAP ([28]) for the computations. Look at the following diagram, which describes all subgroups of Γ of finite index (Γ has no non-trivial normal subgroups of infinite index by Proposition 9).



Here are some explanations: N_1, N_2, N_3, N_4 are normal subgroups of Γ. H_1, H_2, H_3 are subgroups of Γ, but not normal. The index in Γ is given on the left side of the diagram. All arrows are inclusions. The subgroups of Γ are defined as follows:
N_1 is the kernel of $\Gamma \to S_2$ given by $a_i \mapsto ()$, $b_j \mapsto (1,2)$.
N_2 is the kernel of $\Gamma \to S_2$ given by $a_i \mapsto (1,2)$, $b_j \mapsto ()$.
N_3 is the kernel of $\Gamma \to S_2$ given by $a_i \mapsto (1,2)$, $b_j \mapsto (1,2)$.
(By the way, the normal subgroups N_1, N_2, N_3, defined as above, always exist in a $(2m, 2n)$-group Γ.)

N_4 is the kernel of $\Gamma \to S_3$ given by $a_1, a_2 \mapsto (1,2)(3,5)(4,6)$, $a_3, a_4, a_5 \mapsto (1,3)(2,4)(5,6)$, $b_1, b_2, b_3, b_4, b_5 \mapsto ()$, $b_6 \mapsto (1,4,5)(2,3,6)$.
$H_1 := \langle a_1, a_5 a_3^{-1}, b_1 \rangle$.
$H_2 := \langle a_1, a_5 a_3^{-1}, b_2 b_1^{-1} \rangle$.
$H_3 := \langle a_5 a_3^{-1}, b_1 a_1^{-1}, b_2 a_1^{-1} \rangle$.
We have $\Gamma/\Gamma^* \cong D_6$, the dihedral group of order 12; $\Gamma/N_4 \cong S_3$, $N_1/\Gamma^* \cong S_3$, $N_2/\Gamma^* \cong \mathbb{Z}_6$, $N_3/\Gamma^* \cong S_3$, $H_1/\Gamma^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,
$[\Gamma, \Gamma] = [N_1, N_1] = [N_3, N_3] = \Gamma_0$, $[\Gamma_0, \Gamma_0] = [N_2, N_2] = [N_4, N_4] = [H_1, H_1] = [H_2, H_2] = [H_3, H_3] = \Gamma^*$.
The following commutators are not in Γ^*: $[a_1, a_3]$, $[a_1, a_4]$, $[a_1, a_5]$, $[a_1, b_6]$, $[a_2, a_3]$, $[a_2, a_4]$, $[a_2, a_5]$, $[a_2, b_6]$, $[a_3, b_6]$, $[a_4, b_6]$, $[a_5, b_6]$.
In addition, see Table 9 for the orders of some quotients of Γ.

$\lvert\Gamma/\langle\langle w^k \rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1$	2	12	2	12	2	12	2	12	2	12	2	12
a_2	2	12	2	12	2	12	2	12	2	12	2	12
a_3	2	12	2	12	2	12	2	12	2	12	2	12
a_4	2	12	2	12	2	12	2	12	2	12	2	12
a_5	2	12	2	12	2	12	2	12	2	12	2	12
b_1	6	12	6	12	6	12	6	12	6	12	6	12
b_2	6	12	6	12	6	12	6	12	6	12	6	12
b_3	6	12	6	12	6	12	6	12	6	12	6	12
b_4	6	12	6	12	6	12	6	12	6	12	6	12
b_5	6	12	6	12	6	12	6	12	6	12	6	12
b_6	2	4	6	4	2	12	2	4	6	4	2	12

Table 9: Order of $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$, $w \in \{a_1, \ldots, a_5, b_1, \ldots, b_6\}$, $k = 1, \ldots, 12$, in Example 16

□

Example 17.

$$R(5,5) := \left\{ \begin{array}{lllll} \underline{\underline{a_1 b_1 a_2^{-1} b_2^{-1}}}, & \underline{\underline{a_1 b_2 a_1^{-1} b_1^{-1}}}, & \underline{\underline{a_1 b_3 a_2^{-1} b_3^{-1}}}, & a_1 b_4 a_3 b_4, & a_1 b_5 a_1^{-1} b_5^{-1}, \\ a_1 b_4^{-1} a_2 b_4^{-1}, & \underline{\underline{a_1 b_3^{-1} a_2^{-1} b_2}}, & \underline{\underline{a_1 b_1^{-1} a_2^{-1} b_3}}, & \underline{\underline{a_2 b_2 a_2^{-1} b_1^{-1}}}, & a_2 b_4 a_4 b_4, \\ a_2 b_5 a_5^{-1} b_5^{-1}, & a_2 b_5^{-1} a_5^{-1} b_5, & \underline{a_3 b_1 a_4^{-1} b_2^{-1}}, & \underline{a_3 b_2 a_3^{-1} b_1^{-1}}, & \underline{a_3 b_3 a_4^{-1} b_3^{-1}}, \\ a_3 b_5 a_4 b_4^{-1}, & a_3 b_5^{-1} a_4 b_5^{-1}, & a_3 b_4^{-1} a_4 b_5, & \underline{a_3 b_3^{-1} a_4^{-1} b_2}, & \underline{a_3 b_1^{-1} a_4^{-1} b_3}, \\ \underline{a_4 b_2 a_4^{-1} b_1^{-1}}, & a_5 b_1 a_5^{-1} b_3^{-1}, & a_5 b_2 a_5^{-1} b_2^{-1}, & a_5 b_3 a_5^{-1} b_4, & a_5 b_4 a_5^{-1} b_1 \end{array} \right\}.$$

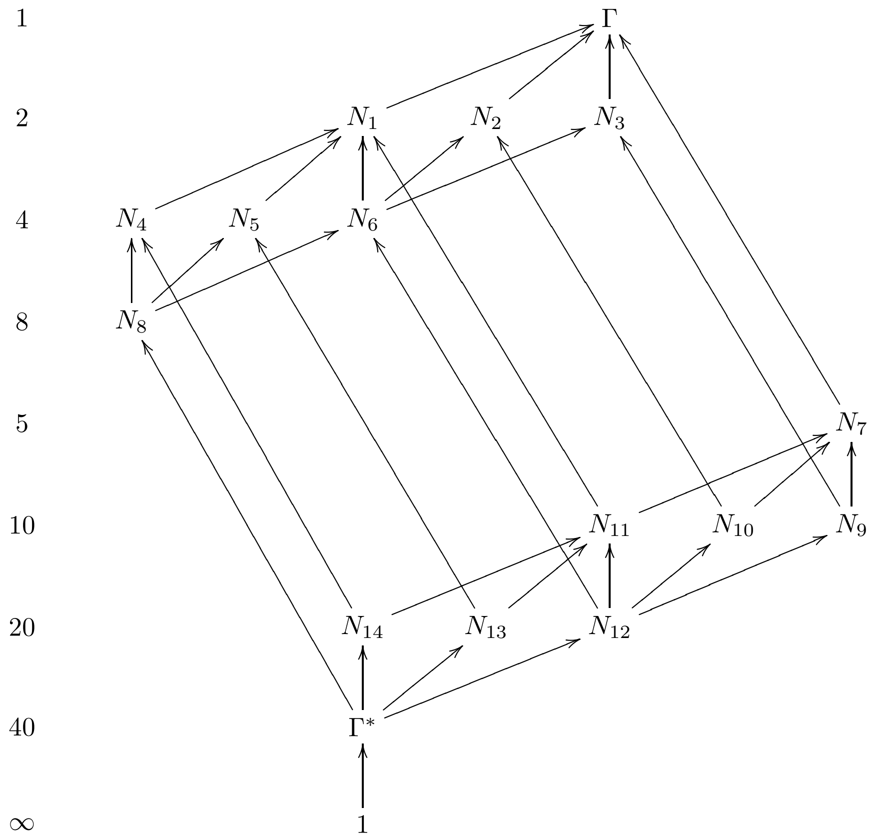
Theorem 17. *Let*

$$\Gamma^* := \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N.$$

(1) $P_h = A_{10}$, $P_v = A_{10}$.

(2) Γ^* *is a finitely presented torsion-free simple group.*

(3) *All finite index subgroups of* Γ *are normal. They are visualized in the following diagram, where all arrows are inclusions.*



Proof. (1)

$$
\begin{aligned}
\rho_v(b_1) &= (7,8)(9,10),\\
\rho_v(b_2) &= (1,2)(3,4),\\
\rho_v(b_3) &= (1,2)(3,4)(7,8)(9,10),\\
\rho_v(b_4) &= (1,9,4,8)(2,10,3,7),\\
\rho_v(b_5) &= (2,5)(3,7)(4,8)(6,9),\\
\\
\rho_h(a_1) &= (1,2)(4,7)(8,10,9),\\
\rho_h(a_2) &= (1,2,3)(4,7)(9,10),\\
\rho_h(a_3) &= (1,2)(4,5,6,7)(8,10,9),\\
\rho_h(a_4) &= (1,2,3)(4,5,6,7)(9,10),\\
\rho_h(a_5) &= (1,7,3)(4,8,10).
\end{aligned}
$$

(2) We apply the same strategy as in the previous theorems.

(3) Using GAP ([28]), we have computed

$$\begin{array}{ll}
N_1 = \langle\!\langle a_1^2, a_1 b_1 \rangle\!\rangle_\Gamma, & \Gamma/N_1 \cong \mathbb{Z}_2 \\
N_2 = \langle\!\langle b_1 \rangle\!\rangle_\Gamma, & \Gamma/N_2 \cong \mathbb{Z}_2 \\
N_3 = \langle\!\langle a_1 \rangle\!\rangle_\Gamma, & \Gamma/N_3 \cong \mathbb{Z}_2 \\
N_4 = \langle\!\langle a_1 b_4 \rangle\!\rangle_\Gamma, & \Gamma/N_4 \cong \mathbb{Z}_4 \\
N_5 = \langle\!\langle a_1 b_5 \rangle\!\rangle_\Gamma, & \Gamma/N_5 \cong \mathbb{Z}_4 \\
N_6 = \langle\!\langle a_1^2 \rangle\!\rangle_\Gamma = \Gamma_0, & \Gamma/N_6 \cong \mathbb{Z}_2^2 \\
N_7 = \langle\!\langle a_1^5, b_1^5 \rangle\!\rangle_\Gamma, & \Gamma/N_7 \cong \mathbb{Z}_5 \\
N_8 = \langle\!\langle a_1^4 \rangle\!\rangle_\Gamma, & \Gamma/N_8 \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \\
N_9 = \langle\!\langle a_1^2 a_3^{-1} \rangle\!\rangle_\Gamma, & \Gamma/N_9 \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \\
N_{10} = \langle\!\langle a_1^2 b_5^{-1} \rangle\!\rangle_\Gamma, & \Gamma/N_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \\
N_{11} = \langle\!\langle a_1^{10}, a_1 b_1 \rangle\!\rangle_\Gamma, & \Gamma/N_{11} \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \\
N_{12} = \langle\!\langle a_1^{10} \rangle\!\rangle_\Gamma, & \Gamma/N_{12} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_5 \\
N_{13} = \langle\!\langle a_1 b_1 \rangle\!\rangle_\Gamma, & \Gamma/N_{13} \cong \mathbb{Z}_4 \times \mathbb{Z}_5 \\
N_{14} = \langle\!\langle b_5 a_3^{-1} \rangle\!\rangle_\Gamma, & \Gamma/N_{14} \cong \mathbb{Z}_4 \times \mathbb{Z}_5 \\
\Gamma^* = [\Gamma, \Gamma] = \langle\!\langle a_1 a_2^{-1} \rangle\!\rangle_\Gamma = \langle\!\langle a_1^{20} \rangle\!\rangle_\Gamma, & \Gamma/\Gamma^* \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5
\end{array}$$

$$\Gamma/\Gamma^* = \Gamma^{ab} \cong \langle a_1, b_5 \mid a_1^2 b_5^{-6}, a_1^6 b_5^2, [a_1, b_5] \rangle.$$

See Table 10 for the orders of some quotients of Γ.

$\lvert\Gamma/\langle\!\langle w^k \rangle\!\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1$	2	4	2	8	10	4	2	8	2	20	2	8
a_2	2	4	2	8	10	4	2	8	2	20	2	8
a_3	2	4	2	8	10	4	2	8	2	20	2	8
a_4	2	4	2	8	10	4	2	8	2	20	2	8
a_5	2	4	2	8	10	4	2	8	2	20	2	8
b_1	2	4	2	8	10	4	2	8	2	20	2	8
b_2	2	4	2	8	10	4	2	8	2	20	2	8
b_3	2	4	2	8	10	4	2	8	2	20	2	8
b_4	2	4	2	8	10	4	2	8	2	20	2	8
b_5	2	4	2	8	10	4	2	8	2	20	2	8

Table 10: Order of $\Gamma/\langle\!\langle w^k \rangle\!\rangle_\Gamma$, $w \in \{a_1, \dots, a_5, b_1, \dots, b_5\}$, $k = 1, \dots, 12$, in Example 17

□

See Appendix C.5 for a list of embeddings of the non-residually finite $(8,6)$-complex of Example 13 into $(10,10)$-complexes X such that P_h and P_v are both primitive permutation groups.

3.8 An example with infinite but no finite quotients

We use an embedding of the non-residually finite $(8,6)$-complex of Example 13 to get a non-simple group $\Gamma_0 < \mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ without proper normal subgroups of finite index.

Example 18.

$$R(5,5) := \left\{ \begin{array}{lllll} \underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_1b_5, & a_1b_5^{-1}a_2b_5^{-1}, \\ a_1b_4^{-1}a_4^{-1}b_4^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}}, & \underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, & a_2b_4a_2b_5, \\ a_2b_4^{-1}a_3^{-1}b_4^{-1}, & \underline{a_3b_1a_4^{-1}b_2^{-1}}, & \underline{a_3b_2a_3^{-1}b_1^{-1}}, & \underline{a_3b_3a_4^{-1}b_3^{-1}}, & a_3b_5a_4b_4^{-1}, \\ a_3b_5^{-1}a_5^{-1}b_5^{-1}, & a_3b_4^{-1}a_4b_5, & \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, & \underline{a_4b_2a_4^{-1}b_1^{-1}}, \\ a_4b_5^{-1}a_5b_5^{-1}, & a_5b_1a_5b_4, & a_5b_2a_5^{-1}b_3, & a_5b_3a_5^{-1}b_2, & a_5b_4^{-1}a_5b_1^{-1} \end{array} \right\}.$$

Theorem 18. (1) *$P_h < S_{10}$ is transitive, but not quasi-primitive; $P_v = S_{10}$.*

(2) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(3) *Γ_0 has no proper normal subgroups of finite index.*

(4) *Γ_0 is not simple.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (5,6)(7,8)(9,10), \\ \rho_v(b_2) &= (1,2)(3,4), \\ \rho_v(b_3) &= (1,2)(3,4)(7,8)(9,10), \\ \rho_v(b_4) &= (1,4,8,9,2,3,7,10)(5,6), \\ \rho_v(b_5) &= (1,9,2,10)(3,5,7)(4,6,8), \end{aligned}$$

generating a transitive subgroup $P_h < S_{10}$ of order 3840. It is not quasi-primitive, since P_h has a normal subgroup of order 2 generated by the element $(1,2)(3,4)(5,6)(7,8)(9,10) = \rho_v(b_1)\rho_v(b_2)$.

$$\begin{aligned} \rho_h(a_1) &= (1,2)(4,7,5,6)(8,10,9), \\ \rho_h(a_2) &= (1,2,3)(4,7,5,6)(9,10), \\ \rho_h(a_3) &= (1,2)(4,5,6,7)(8,10,9), \\ \rho_h(a_4) &= (1,2,3)(4,5,6,7)(9,10), \\ \rho_h(a_5) &= (1,7)(2,8)(3,9)(4,10)(5,6). \end{aligned}$$

(2) Easy computations.

(3) By construction, the non-residually finite complex of Example 13 embeds into X. Set $w := a_2a_1^{-1}a_3a_4^{-1}$. As in Theorem 14, we observe that $\langle\langle w \rangle\rangle_\Gamma = \Gamma_0$, in particular

$$\langle\langle w \rangle\rangle_\Gamma > \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

Since

$$w \in \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N,$$

we conclude that

$$\langle\langle w \rangle\rangle_\Gamma = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N = \Gamma_0.$$

Assume now that M is a finite index normal subgroup of Γ_0. Then M has finite index in Γ and therefore

$$M > \bigcap_{\substack{\text{f.i.} \\ L < \Gamma}} L = \bigcap_{\substack{\text{f.i.} \\ N \lhd \Gamma}} N = \Gamma_0,$$

hence $M = \Gamma_0$.

(4) $\mathrm{QZ}(H_1) \cap \Gamma_0$ is a non-trivial normal subgroup of infinite index in Γ_0. More precisely,

$$A := \{(a_1 a_2^{-1})^{\pm 2}, (a_2^{-1} a_1)^{\pm 2}, (a_3 a_4^{-1})^{\pm 2}, (a_4^{-1} a_3)^{\pm 2}, a_5^{\pm 4}\} \subset \Lambda_1 \cap \Gamma_0 < \mathrm{QZ}(H_1) \cap \Gamma_0,$$

since for each $a \in A$ and $b \in E_v$ we have $\rho_h(a)(b) = b$ and $\rho_v(b)(a) \in A$, using Lemma 26(1a).

Note that for the vertical decomposition of $\Gamma_0 \cong F_9 *_{F_{81}} F_9$, which exists by Proposition 3 because of

$$P_h = \langle \rho_v(b_1^2), \rho_v(b_1 b_2), \rho_v(b_1 b_4), \rho_v(b_5^2) \rangle,$$

we have $|F_{81} \backslash F_9 / F_{81}| = 3$ (> 2 since P_h is not 2-transitive by Proposition 22) and Γ_0 is therefore even SQ-universal, according to Rips' result mentioned in Section 2.2.

$\square$

Remarks. (see Appendix E.5 for much more history)

(1) Graham Higman's group

$$H = \langle a, b, c, d \mid b^{-1} a b = a^2, c^{-1} b c = b^2, d^{-1} c d = c^2, a^{-1} d a = d^2 \rangle$$

introduced in [33], has no proper normal subgroup of finite index. There is another similarity to Γ_0: Using small cancellation theory, Paul Schupp proved in [63] that H is SQ-universal. By the way, H was used to show the existence of a finitely generated infinite simple group (take the quotient of H by a maximal normal subgroup of H), thus answering a question of Aleksandr G. Kuroš ([40]).

(2) Meenaxi Bhattacharjee has constructed in [6] an amalgamated free product $F_3 *_{F_{13}} F_3$ without non-trivial finite quotients. Note that in this group there are non-trivial elements a such that for example a^2 and a^5 are conjugate (cf. Proposition 17).

(3) In [69], Wise gave a construction of a square complex, whose fundamental group has no non-trivial finite quotients.

As usual, we give a table with the orders of some quotients of Γ (Table 11). The infinite quotients correspond to elements in Λ_1.

3.9 $(8, 8)$-group with non-virtually torsion-free quotient

Using an idea of Wise ([69, Section II.6]), we construct a quotient of an $(8, 8)$-group which is not virtually torsion-free.

Lemma 30. *(cf. [69, Easy Lemma II.6.1]) Let G be a non-residually finite group and $g \in G$ an element such that*

$$1 \neq g \in \bigcap_{\substack{f.i. \\ N \lhd G}} N$$

and assume that $g \notin \langle\langle g^n \rangle\rangle_G$ for some $n \geq 2$ (equivalently: $\langle\langle g^n \rangle\rangle_G \subsetneq \langle\langle g \rangle\rangle_G$). Then the quotient $G / \langle\langle g^n \rangle\rangle_G$ is non-residually finite and not virtually torsion-free.

$\lvert\Gamma/\langle\langle w^k\rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w=a_1$	2	4	2	4	2	4	2	4	2	4	2	4
a_2	2	4	2	4	2	4	2	4	2	4	2	4
a_3	2	4	2	4	2	4	2	4	2	4	2	4
a_4	2	4	2	4	2	4	2	4	2	4	2	4
a_5	2	4	2	∞	2	4	2	∞	2	4	2	∞
b_1	2	4	2	4	2	4	2	4	2	4	2	4
b_2	2	4	2	4	2	4	2	4	2	4	2	4
b_3	2	4	2	4	2	4	2	4	2	4	2	4
b_4	2	4	2	4	2	4	2	4	2	4	2	4
b_5	2	4	2	4	2	4	2	4	2	4	2	4

Table 11: Order of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$, $w \in \{a_1,\dots,a_5,b_1,\dots,b_5\}$, $k=1,\dots,12$, in Example 18

Proof. (cf. [69, Proof of Easy Lemma II.6.1]) Let $H < G/\langle\langle g^n\rangle\rangle_G =: Q$ be a subgroup of finite index (say of index k). Let $\psi = \phi \circ \pi$ be the composition homomorphism

$$\psi : G \xrightarrow{\pi} Q \xrightarrow{\phi} S_k,$$

where π is the canonical projection and ϕ is induced by left multiplication on left cosets in Q/H (cf. proof of Lemma 10). Since $\ker\psi \lhd G$ and $[G:\ker\psi] \le |S_k| = k! < \infty$, we have $g \in \ker\psi$, hence $\pi(g) = g\langle\langle g^n\rangle\rangle_G \in \ker\phi < H$. By assumption $g \notin \langle\langle g^n\rangle\rangle_G$, which implies $g\langle\langle g^n\rangle\rangle_G \neq 1_Q$. We conclude that Q is non-residually finite. H is not torsion-free, since $(g\langle\langle g^n\rangle\rangle_G)^n = \langle\langle g^n\rangle\rangle_G = 1_H$. $\square$

Example 19.

$$R(4,4) := \left\{ \begin{array}{llll} \underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_2^{-1}b_4, \\ a_1b_4^{-1}a_2^{-1}b_4^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}}, & \underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, \\ \underline{a_3b_1a_4^{-1}b_2^{-1}}, & a_3b_2a_3^{-1}b_1^{-1}, & a_3b_3a_4^{-1}b_3^{-1}, & a_3b_4a_3^{-1}b_4, \\ \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, & \underline{a_4b_2a_4^{-1}b_1^{-1}}, & a_4b_4a_4^{-1}b_4^{-1} \end{array} \right\}.$$

Theorem 19. *Let $w := a_2a_1^{-1}a_3a_4^{-1}$. Then $\Gamma/\langle\langle w^2\rangle\rangle_\Gamma$ is non-residually finite and not virtually torsion-free. More precisely,*

$$w\langle\langle w^2\rangle\rangle_\Gamma \in \bigcap_{N \overset{f.i.}{\lhd} \Gamma/\langle\langle w^2\rangle\rangle_\Gamma} N < \Gamma/\langle\langle w^2\rangle\rangle_\Gamma$$

has order 2.

Proof. The non-residually finite complex of Example 13 embeds into X and induces a π_1-injection by Proposition 5(1), in particular

$$w \in \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

Note that $w \notin \Lambda_1$, since $\rho_h(w)(b_4) = b_4^{-1} \neq b_4$ (see Figure 7).

However,

$$A := \{w^2, (a_1a_2^{-1}a_4a_3^{-1})^2, (a_1a_2^{-1}a_3a_4^{-1})^2, (a_2a_1^{-1}a_4a_3^{-1})^2\} \subset \Lambda_1,$$

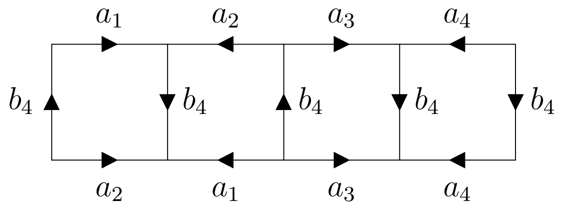


Figure 7: $\rho_h(w)(b_4) = b_4^{-1}$

since for each $a \in A$ and $b \in E_v$ we have $\rho_h(a)(b) = b$ and $\rho_v(b)(a) \in A$. Because of $w^2 \in \Lambda_1$ and $\Lambda_1 \lhd \Gamma$, we conclude that $\langle\langle w^2 \rangle\rangle_\Gamma < \Lambda_1$ and therefore $w \notin \langle\langle w^2 \rangle\rangle_\Gamma$. Now apply Lemma 30 to the quotient $\Gamma/\langle\langle w^2 \rangle\rangle_\Gamma$. $\square$

Remark. Let Γ be a $(2m, 2n)$-group satisfying the normal subgroup theorem (every non-trivial normal subgroup of Γ has finite index, see Proposition 9). Then every quotient of Γ is obviously virtually torsion-free.

3.10 Simplification ?

Recall Example 12 of Wise:

$$R(2,3) := \left\{ \begin{array}{ll} a_1 b_2 a_1^{-1} b_1^{-1}, & a_2 b_2 a_2^{-1} b_1^{-1}, \\ a_1 b_3 a_2^{-1} b_3^{-1}, & a_1 b_1 a_2^{-1} b_2^{-1}, \\ a_2 b_1 a_1^{-1} b_3^{-1}, & a_2 b_3 a_1^{-1} b_2^{-1} \end{array} \right\}.$$

Lemma 31. *Let $\theta : \Gamma \to \Gamma$, $\gamma \mapsto b_3 \gamma b_3^{-1}$ be the conjugation by b_3. Then $\mathrm{Fix}(\theta) = \langle b_3 \rangle$.*

Proof. Note that $\mathrm{Fix}(\theta) = \{\gamma \in \Gamma : b_3 \gamma b_3^{-1} = \gamma\}$ is the centralizer of b_3 in Γ. The statement follows now from Proposition 8(1b). $\square$

Lemma 32. $[\Gamma, \Gamma] = \langle\langle a_1 a_2^{-1} \rangle\rangle_\Gamma$ *and* $\Gamma/[\Gamma, \Gamma] \cong \langle a_1, b_1 \mid a_1 b_1 = b_1 a_1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Proof. $[\Gamma, \Gamma] > \langle\langle a_1 a_2^{-1} \rangle\rangle_\Gamma$, since $a_1 a_2^{-1} = [a_1, b_3^{-1}] \in [\Gamma, \Gamma]$.
Let $N \lhd \Gamma$ be a normal subgroup containing $a_1 a_2^{-1}$ (e.g. $N = \langle\langle a_1 a_2^{-1} \rangle\rangle_\Gamma$). Then $a_1 N = a_2 N$, hence $a_1 b_2 N = b_1 a_1 N$, $a_1 b_3 N = b_3 a_1 N$, $a_1 b_1 N = b_2 a_1 N$, $a_1 b_1 N = b_3 a_1 N$, $a_1 b_3 N = b_2 a_1 N$, i.e. $b_1 N = b_2 N = b_3 N$ and $a_1 b_1 N = b_1 a_1 N$. In particular, Γ/N is generated by $\{a_1 N, b_1 N\}$ and abelian, therefore $[\Gamma, \Gamma] < N$. $\square$

Conjecture 13. Γ *is not $\langle b_3 \rangle$-separable. More specifically, $a_1 a_2^{-1}$ is not separable from $\langle b_3 \rangle$ in any finite quotient of* Γ.

See Lemma 34 for a possible proof of Conjecture 13. As a "corollary" we would have

Conjecture 14. Γ *is non-residually finite with*

$$(a_1 a_2^{-1})^{-1}\, \theta(a_1 a_2^{-1}) = [a_2 a_1^{-1}, b_3] \in \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N.$$

"Proof". This would directly follow from Lemma 27 using Conjecture 13 and Lemma 31. $\square$

Conjecture 15. *Take*

$$R(4,4) := \left\{ \begin{array}{llll} \underline{a_1b_1a_2^{-1}b_2^{-1}}, & \underline{a_1b_2a_1^{-1}b_1^{-1}}, & \underline{a_1b_3a_2^{-1}b_3^{-1}}, & a_1b_4a_4^{-1}b_4, \\ & & & \\ a_1b_4^{-1}a_2b_4^{-1}, & \underline{a_1b_3^{-1}a_2^{-1}b_2}, & \underline{a_1b_1^{-1}a_2^{-1}b_3}, & \underline{a_2b_2a_2^{-1}b_1^{-1}}, \\ & & & \\ a_2b_4a_3b_4, & a_3b_1a_3b_2, & a_3b_3a_4^{-1}b_3^{-1}, & a_3b_4^{-1}a_4^{-1}b_3, \\ & & & \\ a_3b_3^{-1}a_4^{-1}b_2^{-1}, & a_3b_2^{-1}a_4^{-1}b_4^{-1}, & a_3b_1^{-1}a_4b_1^{-1}, & a_4b_1a_4b_2^{-1} \end{array} \right\}.$$

Then the corresponding $\Gamma_0 \cong F_7 *_{F_{49}} F_7$ is a finitely presented torsion-free simple group.

"Proof". Using Conjecture 14, this would follow as in Section 3.6, because $P_h = A_8$, $P_v = A_8$, the complex of Example 12 is embedded in X, and $\langle\langle [a_2a_1^{-1}, b_3] \rangle\rangle_\Gamma = \Gamma_0$. $\square$

We return to Example 12 of Wise:

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_2a_1^{-1}b_1^{-1}, & a_2b_2a_2^{-1}b_1^{-1}, \\ & \\ a_1b_3a_2^{-1}b_3^{-1}, & a_1b_1a_2^{-1}b_2^{-1}, \\ & \\ a_2b_1a_1^{-1}b_3^{-1}, & a_2b_3a_1^{-1}b_2^{-1} \end{array} \right\}.$$

Lemma 33. $\langle\langle [a_2a_1^{-1}, b_3] \rangle\rangle_\Gamma = \langle\langle a_1a_2^{-1} \rangle\rangle_\Gamma$.

Proof. We have checked it using MAGNUS ([49]). The direction $\langle\langle [a_2a_1^{-1}, b_3] \rangle\rangle_\Gamma < \langle\langle a_1a_2^{-1} \rangle\rangle_\Gamma$ is obvious, since $[a_2a_1^{-1}, b_3] \in [\Gamma, \Gamma] = \langle\langle a_1a_2^{-1} \rangle\rangle_\Gamma$ (by Lemma 32). $\square$

Conjecture 16.

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = [\Gamma, \Gamma].$$

"Partial Proof". By Lemma 32, Lemma 33 and Conjecture 14, we have

$$[\Gamma, \Gamma] = \langle\langle a_1a_2^{-1} \rangle\rangle_\Gamma = \langle\langle [a_2a_1^{-1}, b_3] \rangle\rangle_\Gamma < \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

$\square$

How to prove Conjecture 13 ?

Lemma 34. *Suppose that there is an element $w \in \Gamma \setminus \langle b_3 \rangle$ such that for each $k \in \mathbb{N}$ we have $w \in \langle b_3 \rangle \langle\langle b_3^{2k} \rangle\rangle_\Gamma$. Then Γ is not $\langle b_3 \rangle$-separable.*

Proof. Let $\psi : \Gamma \to Q$ be a homomorphism onto a finite group Q and let k be the order of $\psi(b_3)$ in Q. Then ψ is the composition of

$$\Gamma \xrightarrow{\psi_1} \Gamma / \langle\langle b_3^{2k} \rangle\rangle_\Gamma \xrightarrow{\psi_2} \Gamma / \langle\langle b_3^{k} \rangle\rangle_\Gamma \xrightarrow{\psi_3} Q.$$

Hence

$$\psi(w) = \psi_3\psi_2(w\langle\langle b_3^{2k} \rangle\rangle_\Gamma) \in \psi_3\psi_2(\langle b_3 \rangle \langle\langle b_3^{2k} \rangle\rangle_\Gamma) = \psi(\langle b_3 \rangle)$$

and Γ is not $\langle b_3 \rangle$-separable. $\square$

Question 4. *Is it true that $a_1a_2^{-1} \in \langle b_3 \rangle \langle\langle b_3^{2k} \rangle\rangle_\Gamma$ for all $k \in \mathbb{N}$?*

Remarks. (1) $\langle\langle b_3^i \rangle\rangle_\Gamma \neq \langle\langle b_3^j \rangle\rangle_\Gamma$, if $i \neq j$ and $i, j \in \mathbb{N}$, since $(\Gamma / \langle\langle b_3^i \rangle\rangle_\Gamma)^{ab} \cong \mathbb{Z} \times \mathbb{Z}_i$.

(2) $a_1a_2^{-1} \in \langle\langle b_3^{2k} \rangle\rangle_\Gamma$ if and only if $\Gamma / \langle\langle b_3^{2k} \rangle\rangle_\Gamma$ is abelian. This follows from Lemma 32. Using MAGNUS ([49]): $\Gamma / \langle\langle b_3^{8} \rangle\rangle_\Gamma$ is not abelian, in other words $a_1a_2^{-1} \notin \langle\langle b_3^{8} \rangle\rangle_\Gamma$.

3.11 Project: simple and property (T)

Assumption. Let Γ_D be the non-residually finite group of Theorem 13 and $w := a_2 a_1^{-1} a_3 a_4^{-1}$. We assume in this section that Γ_D embeds either in a (A_{2m}, P_v)-group Γ such that P_v is quasi-primitive, not 2-transitive, or in a (P_h, A_{2n})-group Γ such that P_h is quasi-primitive, not 2-transitive. Moreover, we assume that $[\Gamma : \langle\langle w \rangle\rangle_\Gamma] < \infty$ (the easiest case would be $\langle\langle w \rangle\rangle_\Gamma = \Gamma_0$) and that $\mathrm{pr}_i(\langle\langle w \rangle\rangle_\Gamma)$ is locally quasi-primitive, $i = 1, 2$.

Lemma 35. *Let G be a finitely generated group and $N \lhd G$ ($N \neq G$) a proper normal subgroup. Then there is a maximal proper normal subgroup M of G containing N, i.e. $N < M \lhd G$, $M \neq G$, and $M \lneqq \tilde{M} \lhd G$ always implies $\tilde{M} = G$.*

Proof. Let $G = \{g_0, g_1, \ldots\}$ be an enumeration of G. We define $N_0 := N$ and for $k = 1, 2, 3, \ldots$

$$N_k := \begin{cases} N_{k-1}, & \text{if } \langle\langle N_{k-1}, g_{k-1} \rangle\rangle_G = G \\ \langle\langle N_{k-1}, g_{k-1} \rangle\rangle_G, & \text{if } \langle\langle N_{k-1}, g_{k-1} \rangle\rangle_G \neq G \end{cases}$$

and finally

$$M := \bigcup_{k \in \mathbb{N}_0} N_k.$$

Then obviously M is a normal subgroup of G containing N. Let $\{h_1, \ldots, h_l\}$ be a finite set generating G. Assume that $M = G$. Then $h_i \in M$, $i = 1, \ldots, l$, hence $h_i \in N_{k_i}$ for some k_i and $\{h_1, \ldots, h_l\} \subset N_{\max\{k_i\}_{i=1,\ldots,l}}$. But this implies $N_{\max\{k_i\}_{i=1,\ldots,l}} = G$, which contradicts the construction, thus $M \neq G$. It remains to show that M is "maximal". Let $\tilde{M}$ be a group such that $M \lneqq \tilde{M} \lhd G$. Take an element $\tilde{g} \in \tilde{M} \setminus M$. Then $\tilde{g} = g_j$ for some $j \in \mathbb{N}_0$. We have $g_j \notin M$, in particular $g_j \notin N_{j+1}$, hence by definition of N_{j+1}: $\langle\langle N_j, g_j \rangle\rangle_G = G$. But $\tilde{M} \lhd G$ contains N_j and g_j, therefore $\tilde{M} = G$. □

Remark. We do not know a reference for this well-known lemma, but Bernhard H. Neumann showed in [54, Theorem (5)] that in a finitely generated group G every proper subgroup of G is contained in a maximal proper subgroup of G.

Proposition 36. *Let N be a non-trivial proper normal subgroup of $\langle\langle w \rangle\rangle_\Gamma$. Let M be a maximal proper normal subgroup of $\langle\langle w \rangle\rangle_\Gamma$ containing N (use Lemma 35 to guarantee the existence of M). If $\overline{\mathrm{pr}_i(M)} \not\leq \mathrm{QZ}(\overline{\mathrm{pr}_i(\langle\langle w \rangle\rangle_\Gamma)})$, $i = 1, 2$, then $\langle\langle w \rangle\rangle_\Gamma / M$ is an infinite simple group having property (T).*

Proof. First note that

$$\langle\langle w \rangle\rangle_\Gamma = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N$$

and $\langle\langle w \rangle\rangle_\Gamma$ has no proper subgroups of finite index, hence $\langle\langle w \rangle\rangle_\Gamma / M$ is infinite. This group is simple because of the maximality of M. To show that $\langle\langle w \rangle\rangle_\Gamma / M$ has property (T), apply [16, Proposition 3.1], using [15, Proposition 1.2.1] and [15, Lemma 1.4.2]. □

4 More examples of $(2m, 2n)$-complexes

This section contains a collection of various $(2m, 2n)$-complexes, illustrating some additional interesting phenomena not discussed in Section 2 and 3. For instance, we try to examine the relation between local properties of Γ and (ir)reducibility of Γ.

4.1 Local groups vs. irreducibility

We start with two examples which look very similar, because they have for example exactly the same local groups P_h and P_v, but Example 20 is irreducible, whereas Example 21 is reducible. Moreover, Example 20 exhibits some other remarkable properties.

Example 20.

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2^{-1}, \\ \\ a_1b_3^{-1}a_2b_2, & a_2b_1a_3^{-1}b_1^{-1}, & a_2b_3a_2b_2^{-1}, \\ \\ a_2b_1^{-1}a_3^{-1}b_1, & a_3b_2a_3^{-1}b_2^{-1}, & a_3b_3a_3^{-1}b_3^{-1} \end{array} \right\}.$$

Theorem 20. (1) *$P_h = A_6$, $P_v \cong \mathbb{Z}_2 < S_6$ and Γ is irreducible.*

(2) *$H_2(x_v)$ is a pro-2 group.*

(3) *$\Lambda_2 \neq 1$, in particular $\mathrm{QZ}(H_2) \neq 1$.*

(4) *$\langle a_1, a_2, a_3\rangle \cong \mathrm{pr}_2(\langle a_1, a_2, a_3\rangle) \cong \mathrm{pr}_2(\langle a_1, a_2, a_3\rangle)(x_v) \cong \mathrm{pr}_2(\Gamma)(x_v) < \mathrm{Aut}(\mathcal{T}_{2n})(x_v)$ stabilizes pointwise a bi-infinite geodesic in $\mathcal{T}_{2n} = \mathcal{T}_6$ through x_v.*

(5) *$\Gamma^{ab} \cong \mathbb{Z}^2 \times \mathbb{Z}_2$, in particular it is an infinite group.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (1,2,5), \\ \rho_v(b_3) &= (2,6,5), \\ \\ \rho_h(a_1) &= (2,3)(4,5), \\ \rho_h(a_2) &= (2,3)(4,5), \\ \rho_h(a_3) &= (). \end{aligned}$$

To see that Γ is irreducible, compute $|P_h^{(2)}| = 360 \cdot 60^6$.

(2) This follows directly from the subsequent Proposition 37.

(3) $\{b_1^2, b_2^3, b_3^3\} \subset \Lambda_2$. For example $b_1^2 \in \Lambda_2$, since b_1^2 commutes with each element in E_h (see Lemma 26(1b)). Note that Λ_2 is a normal subgroup of $\langle b_1, \dots, b_n\rangle$ of infinite index, since Γ is irreducible. In particular, Λ_2 is a non-finitely generated free normal subgroup of Γ.

(4) The map pr_2 is injective because $\mathrm{QZ}(H_1) = 1$ by Proposition [15, Proposition 3.1.2]. This gives the first claimed isomorphism. The two other isomorphisms are based on the identification

$$\langle a_1, a_2, a_3\rangle \cong \{\gamma \in \Gamma : \mathrm{pr}_2(\gamma)(x_v) = x_v\}$$

proved in [16, Chapter 1]. Since $\rho_h(a)(b_1) = b_1$ for each $a \in E_h$, the bi-infinite geodesic $(b_1^k)_{k\in\mathbb{Z}}$ through x_v is fixed.

(5) This is an easy computation.

□

Proposition 37. *Let Γ be a (P_h, P_v)-group with $|P_v| = 2$. Then $H_2(x_v)$ is a pro-2 group (an infinite group if and only if Γ is irreducible).*

Proof. Consider the commutative diagram, where p_k is the restriction map, $k \in \mathbb{N}$.

$$\begin{tikzcd}
\langle a_1, \dots, a_m \rangle \arrow[r, two heads, "\rho_h^{(k+1)}"] \arrow[dr, two heads, "\rho_h^{(k)}"'] & P_v^{(k+1)} < \mathrm{Sym}(E_v^{(k+1)}) \arrow[d, "p_k"] \\
& P_v^{(k)} < \mathrm{Sym}(E_v^{(k)})
\end{tikzcd}$$

We want to show that $P_v^{(k)}$ is a 2-group for each $k \in \mathbb{N}$. Since $|P_v| = 2$ and $P_v^{(k)} \cong P_v^{(k+1)}/\ker(p_k)$, it is enough to show that $\ker(p_k)$ is a 2-group (or trivial). This follows, if any element $\sigma \in \ker(p_k)$ has order 1 or 2 in $P_v^{(k+1)}$. Given $\sigma \in \ker(p_k)$, write $\sigma = \rho_h^{(k+1)}(a)$ for an appropriate element a in $\langle a_1, \dots, a_m \rangle$. Let b be any element in $E_v^{(k+1)}$. Decompose $b = b' \cdot b''$, with $b' \in E_v^{(k)}$, $b'' \in E_v$ and define $\tilde{a} := \rho_v^{(|a|)}(b')(a)$ (see Figure 8). Then

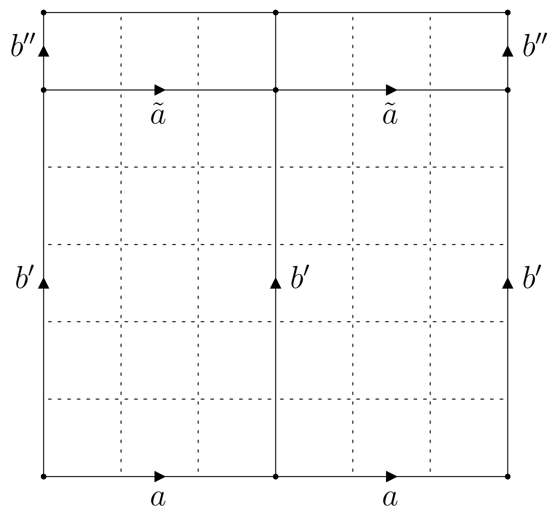


Figure 8: Illustration in the proof of Proposition 37

$$\sigma^2(b) = \rho_h^{(k+1)}(a^2)(b' \cdot b'') \overset{(\star)}{=} b' \cdot \rho_h(\tilde{a}^2)(b'') \overset{(\star\star)}{=} b' \cdot b'' = b,$$

where the equation $(\star)$ uses the commutativity of the diagram above and $(\star\star)$ follows from the assumption $|P_v| = 2$. □

The following conjecture is true at least for $k \le 6$, i.e. we have computed $|P_v^{(2)}| = 4$, $|P_v^{(3)}| = 16$, $|P_v^{(4)}| = 32$, $|P_v^{(5)}| = 128$, $|P_v^{(6)}| = 256$.

Conjecture 17. *For Γ defined in Example 20 and $l \in \mathbb{N}$*

$$|P_v^{(k)}| = \begin{cases} 2^{3l-1}, & k = 2l \\ 2^{3l-2}, & k = 2l-1. \end{cases}$$

The next example has seven (of nine) geometric squares in common with Example 20. The two different geometric squares are underlined. They can be obtained from the corresponding two geometric squares $a_3b_2a_3^{-1}b_2^{-1}$, $a_3b_3a_3^{-1}b_3^{-1}$ in Example 20 by a single "surgery" operation indicated in Figure 9. For a more general description of surgery techniques in square complexes, see [16, Chapter 6.2.2].

Example 21.

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2^{-1}, \\ \\ a_1b_3^{-1}a_2b_2, & a_2b_1a_3^{-1}b_1^{-1}, & a_2b_3a_2b_2^{-1}, \\ \\ a_2b_1^{-1}a_3^{-1}b_1, & \underline{a_3b_2a_3^{-1}b_3^{-1}}, & \underline{a_3b_3a_3^{-1}b_2^{-1}} \end{array} \right\}.$$

Theorem 21. (1) *$P_h = A_6$, $P_v \cong \mathbb{Z}_2 < S_6$, but Γ is reducible.*

(2) *In general, it is not possible to determine whether a given $(2m, 2n)$-group is reducible or irreducible only by knowing the local groups P_h and P_v.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (1,2,5), \\ \rho_v(b_3) &= (2,6,5), \\ \\ \rho_h(a_1) &= (2,3)(4,5), \\ \rho_h(a_2) &= (2,3)(4,5), \\ \rho_h(a_3) &= (2,3)(4,5). \end{aligned}$$

We compute $|P_h^{(2)}| = 360 = |P_h|$, hence Γ is reducible by Proposition 1(2a). Observe that $|P_v^{(k)}| = 2$ for all $k \in \mathbb{N}$.

(2) Example 20 and Example 21 have exactly the same local groups P_h and P_v, but Example 20 is irreducible, whereas Example 21 is reducible.

□

4.2 Another small irreducible example with $\mathrm{QZ} \neq 1$

We present another example with $\mathrm{QZ}(H_2) \neq 1$, but in addition to Example 20, P_v will be transitive.

Example 22.

$$R(3,2) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_2^{-1}a_3^{-1}b_1, \\ \\ a_2b_1a_3b_1, & a_2b_2a_2b_1^{-1}, & a_2b_2^{-1}a_3b_2^{-1} \end{array} \right\}.$$

Theorem 22. (1) *$P_h \cong A_6$, $P_v \cong D_4 < S_4$ is transitive.*

(2) *Γ is irreducible.*

(3) $\mathrm{QZ}(H_2) \neq 1$.

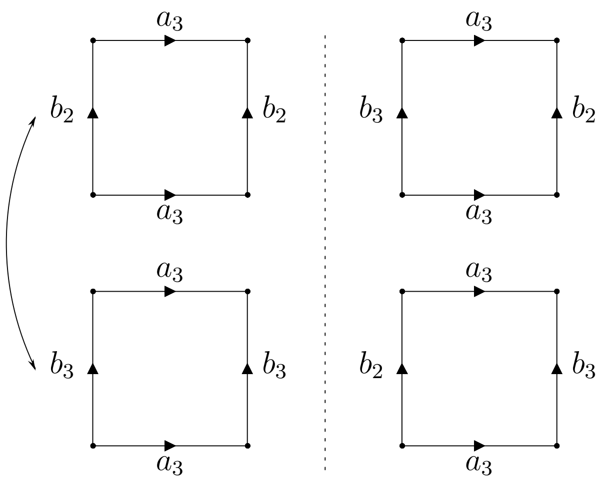


Figure 9: "Surgery" on Example 20 (on the left) to get Example 21 (on the right)

Proof. (1)

$$\rho_v(b_1) = (1, 4, 2, 5, 3),$$
$$\rho_v(b_2) = (2, 4, 6, 3, 5),$$

$$\rho_h(a_1) = (1, 2)(3, 4),$$
$$\rho_h(a_2) = (1, 2, 3, 4),$$
$$\rho_h(a_3) = (1, 2, 3, 4).$$

(2) Γ is irreducible by Proposition 1(1a), since $|P_h^{(2)}| = 360 \cdot 60^6$.

(3) Using Lemma 26(1b), $B := \{(b_1b_2)^3, (b_2b_1)^3, (b_1b_2)^{-3}, (b_2b_1)^{-3}\} \subset \Lambda_2$, since for each $b \in B$ and $a \in E_h$ we have $\rho_v(b)(a) = a$ and $\rho_h(a)(b) \in B$.

□

4.3 Reducible but $|P_h^{(3)}| < |P_h^{(4)}|$

We construct now a reducible lattice $\Gamma < \mathrm{Aut}(\mathcal{T}_4) \times \mathrm{Aut}(\mathcal{T}_6)$, where $P_h^{(k)}$ "stops late".

Example 23.

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2, \\ a_1b_3a_1b_3^{-1}, & a_2b_1a_2b_2^{-1}, \\ a_2b_2a_2b_3^{-1}, & a_2b_3a_2b_1^{-1} \end{array} \right\}.$$

Theorem 23. (1) *$P_h \cong \mathbb{Z}_2^2 < S_4$, $P_v \cong \mathbb{Z}_2 \times A_4 < S_6$.*

(2) *Γ is reducible, but $|P_h| < |P_h^{(2)}| < |P_h^{(3)}| < |P_h^{(4)}|$.*

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (2,3),\\
\rho_v(b_2) &= (2,3),\\
\rho_v(b_3) &= (1,4)(2,3),\\
\\
\rho_h(a_1) &= (2,5),\\
\rho_h(a_2) &= (1,3,2)(4,6,5).
\end{aligned}$$

(2) Γ is reducible, since $|P_v| = |P_v^{(2)}| = 24$. A computation gives $|P_h^{(2)}| = 8$, $|P_h^{(3)}| = 16$, $|P_h^{(4)}| = 32\,(= |P_h^{(5)}|)$.

□

4.4 Reducible but $|P_h| < |P_h^{(2)}|$ and $|P_v| < |P_v^{(2)}|$

We give a reducible $(4,6)$-complex such that $|P_h| < |P_h^{(2)}|$ and $|P_v| < |P_v^{(2)}|$.

Example 24.

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, \\ \\ a_1b_3a_1^{-1}b_2, & a_2b_1a_2^{-1}b_2^{-1}, \\ \\ a_2b_2a_2^{-1}b_1, & a_2b_3a_2b_3^{-1} \end{array} \right\}.$$

Theorem 24. (1) *$|P_h| < |P_h^{(2)}|$ and $|P_v| < |P_v^{(2)}|$.*

(2) *Γ is reducible.*

Proof. (1) We compute $|P_h| = 2$, $|P_h^{(2)}| = 4$, $|P_v| = 24$, $|P_v^{(2)}| = 48$.

(2) This follows from $|P_v^{(3)}| = 48 = |P_v^{(2)}|$. Note that $|P_h^{(3)}| = |P_h^{(4)}| = 8$.

□

4.5 Local transitivity vs. reducibility

The next two examples shall illustrate, that there is no obvious connection between reducibility and local transitivity.

Example 25.

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, \\ \\ a_1b_3a_2^{-1}b_2^{-1}, & a_1b_3^{-1}a_2^{-1}b_2, \\ \\ a_2b_1a_2^{-1}b_3^{-1}, & a_2b_2a_2^{-1}b_1^{-1} \end{array} \right\}.$$

Theorem 25. (1) *$P_h \cong \mathbb{Z}_2^2 < S_4$, $P_v \cong S_3 < S_6$, both are not transitive.*

(2)

	$\lvert P_h^{(k)}\rvert$	$\lvert P_v^{(k)}\rvert$
$k=1$	4	6
$k=2$	32	48
$k=3$	128	192
$k=4$	1024	1536
$k=5$	8192	12288
$k=6$	65536	98304
...	...	...

Proof. (1)

$$\begin{aligned}\rho_v(b_1) &= (),\\ \rho_v(b_2) &= (1,2),\\ \rho_v(b_3) &= (3,4),\\ \\ \rho_h(a_1) &= (2,3)(4,5),\\ \rho_h(a_2) &= (1,2,3)(4,6,5).\end{aligned}$$

(2) Computation.

□

Conjecture 18. *Let Γ be as defined in Example 25. Then*

(1) $\rho_v((b_1b_2)^{2^k})(w_k(a)) = w_k(a)$, *where $w_k(a)$ is any reduced word in $\langle a_1, a_2\rangle$ of length $k \in \mathbb{N}$.*

(2) $\rho_v((b_1b_2)^{2^k})(a_1^{k+1}) = a_1^k a_2$ *for $k \in \mathbb{N}$.*

It would directly follow from this conjecture that Γ is irreducible.

Remark. It is easy to construct an irreducible (P_h, P_v)-group with P_h and P_v both not being transitive. For example embed any irreducible $(2m, 2n)$-complex into a $(2m+2, 2n+2)$-complex by adding the $m+n+1$ geometric tori $a_1b_{n+1}a_1^{-1}b_{n+1}^{-1}, \dots, a_mb_{n+1}a_m^{-1}b_{n+1}^{-1}, a_{m+1}b_1a_{m+1}^{-1}b_1^{-1}, \dots, a_{m+1}b_na_{m+1}^{-1}b_n^{-1}, a_{m+1}b_{n+1}a_{m+1}^{-1}b_{n+1}^{-1}$ and apply Proposition 5(3). See Example 27 for an explicit realization of this idea.

Conversely, we have the following example:

Example 26.

$$R(2,2) := \left\{ \begin{array}{ll} a_1b_1a_2^{-1}b_1, & a_1b_2a_2^{-1}b_2, \\ \\ a_1b_2^{-1}a_1b_1^{-1}, & a_2b_1a_2b_2 \end{array} \right\}.$$

Theorem 26. *P_h and P_v are transitive, but Γ is reducible.*

Proof.

$$\begin{aligned}\rho_v(b_1) &= (1,4,3,2),\\ \rho_v(b_2) &= (1,4,3,2),\\ \\ \rho_h(a_1) &= (1,3,2,4),\\ \rho_h(a_2) &= (1,4,2,3).\end{aligned}$$

Γ is reducible, since $|P_h^{(2)}| = |P_h| = 4$. □

Question 5. *Is there a reducible (P_h, P_v)-group Γ with P_h transitive and P_v 2-transitive?*

Question 6. *Is there a reducible (P_h, P_v)-group Γ with P_h transitive and P_v primitive?*

Question 7. *Is there a reducible (P_h, P_v)-group Γ with P_h transitive and P_v quasi-primitive?*

4.6 Irreducible but $\Lambda_1 \neq 1$, $\Lambda_2 \neq 1$

It follows from [16, Proposition 1.2] that any reducible $(2m, 2n)$-complex satisfies $\Lambda_1 \neq 1$ and $\Lambda_2 \neq 1$. Embedding the irreducible complex of Example 22, we construct now an *irreducible* $(8, 6)$-complex such that $\Lambda_1 \neq 1 \neq \Lambda_2$.

Example 27.

$$R(4,3) := \left\{ \begin{array}{llll} \underline{a_1 b_1 a_1^{-1} b_2^{-1}}, & \underline{a_1 b_2 a_3 b_1^{-1}}, & a_1 b_3 a_1^{-1} b_3^{-1}, & \underline{a_1 b_2^{-1} a_3^{-1} b_1}, \\ \underline{a_2 b_1 a_3 b_1}, & \underline{a_2 b_2 a_2 b_1^{-1}}, & a_2 b_3 a_2^{-1} b_3^{-1}, & \underline{a_2 b_2^{-1} a_3 b_2^{-1}}, \\ a_3 b_3 a_3^{-1} b_3^{-1}, & a_4 b_1 a_4^{-1} b_1^{-1}, & a_4 b_2 a_4^{-1} b_2^{-1}, & a_4 b_3 a_4^{-1} b_3^{-1} \end{array} \right\}.$$

Theorem 27. (1) *P_h, P_v are not transitive.*

(2) *Γ is irreducible.*

(3) *$\Lambda_1 \neq 1 \neq \Lambda_2$.*

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (1,6,2,7,3), \\ \rho_v(b_2) &= (2,6,8,3,7), \\ \rho_v(b_3) &= (), \\ \\ \rho_h(a_1) &= (1,2)(5,6), \\ \rho_h(a_2) &= (1,2,5,6), \\ \rho_h(a_3) &= (1,2,5,6), \\ \rho_h(a_4) &= (). \end{aligned}$$

(2) The irreducible complex of Example 22 is embedded in X, indicated by the six underlined relators. Now apply Proposition 5(3).

(3) $a_4 \in \Lambda_1$, $b_3 \in \Lambda_2$, applying Lemma 26(1b).

□

4.7 Local groups and transitivity

Examples 28 and 29 have pairwise isomorphic local permutation groups P_h and P_v, but on both sides different transitivity properties. The reason for this is that they are isomorphic but not permutation isomorphic.

Example 28.

$$R(2,3) := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_3^{-1}, \\ a_1 b_3 a_2^{-1} b_1^{-1}, & a_1 b_3^{-1} a_2^{-1} b_1, \\ a_1 b_2^{-1} a_2^{-1} b_3, & a_2 b_1 a_2^{-1} b_2 \end{array} \right\}.$$

Theorem 28. *$P_h \cong \mathbb{Z}_2^2 < S_4$ is not transitive, $P_v \cong \mathbb{Z}_2 \times A_4 < S_6$ is transitive.*

Proof.

$$\begin{aligned}
\rho_v(b_1) &= (1,2),\\
\rho_v(b_2) &= (3,4),\\
\rho_v(b_3) &= (1,2)(3,4),\\
\\
\rho_h(a_1) &= (1,3,2)(4,5,6),\\
\rho_h(a_2) &= (1,3,2,6,4,5).
\end{aligned}$$

□

Example 29.

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_1a_2^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_2, \\ \\ a_1b_3a_2b_3, & a_1b_3^{-1}a_2b_3^{-1}, \\ \\ a_1b_2^{-1}a_2^{-1}b_1^{-1}, & a_1b_1^{-1}a_2^{-1}b_1 \end{array} \right\}.$$

Theorem 29. *$P_h \cong \mathbb{Z}_2^2 < S_4$ is transitive, $P_v \cong \mathbb{Z}_2 \times A_4 < S_6$ is not transitive.*

Proof.

$$\begin{aligned}
\rho_v(b_1) &= (1,2)(3,4),\\
\rho_v(b_2) &= (1,2)(3,4),\\
\rho_v(b_3) &= (1,3)(2,4),\\
\\
\rho_h(a_1) &= (1,5,2)(3,4),\\
\rho_h(a_2) &= (2,5,6)(3,4).
\end{aligned}$$

□

4.8 Irreducibility and finite abelianization

A naive conjecture could be: Γ is irreducible if and only if Γ^{ab} is finite. In Theorem 20 we have seen that one direction of this conjecture is false, i.e. there is an irreducible Γ with infinite abelianization Γ^{ab}. The other direction is false by the following example:

Example 30.

$$R(2,2) := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_1, & a_1b_2a_1b_2^{-1}, \\ \\ a_2b_1a_2b_1^{-1}, & a_2b_2a_2^{-1}b_2 \end{array} \right\}.$$

Theorem 30. *Γ is reducible, but Γ^{ab} is finite.*

Proof. $|P_h| = |P_h^{(2)}| = 4$ shows that Γ is reducible. A simple computation gives $\Gamma^{ab} \cong \mathbb{Z}_2^4$ of order 16. □

Remark. If we add to the non-residually finite $(4,12)$-complex of Example 8 the two geometric tori $a_1b_7a_1^{-1}b_7^{-1}$, $a_2b_7a_2^{-1}b_7^{-1}$, we even get a non-residually finite $(4,14)$-group Γ with infinite abelianization Γ^{ab}. Also Example 13 has this property.

Question 8. *Is there a $(2m,2n)$-group Γ such that P_h, P_v are transitive and Γ^{ab} is infinite?*

4.9 Applying a criterion for irreducibility and non-linearity from [16]

We give here examples of small irreducible non-linear $(2m, 2n)$-groups Γ, where both P_h and P_v are not alternating groups, applying results from [16].

Definition. Let x_h be any vertex in $\mathcal{T}_{2m}$ and

$$P_h^{(2)} \cong H_1(x_h)/H_1(S(x_h, 2)) < \mathrm{Sym}(S(x_h, 2))$$

as introduced in Section 1. Let y_h be any neighbouring vertex of x_h. Then we define (see [16, Chapter 1])

$$K_h = \mathrm{Stab}_{P_h^{(2)}}(S(x_h, 1) \cup S(y_h, 1)).$$

In our applications, this definition is independent of the choice of y_h. See Appendix D.4 for the GAP-program ([28]) computing K_h for $m = 3$. Analogously, one defines the group $K_v < P_v^{(2)}$.

The following result is taken from [16]:

Proposition 38. *([16, Proposition 1.3, Theorem 1.4]) Let Γ be a $(2m, 2n)$-group such that P_h and P_v are primitive permutation groups. If either K_h or K_v is not a p-group, then Γ is irreducible and not linear over any field.*

Remark. There is no $(4, 4)$-group satisfying the assumptions of Proposition 38.

Remark. If $m \geq 3$ and Γ is an irreducible (A_{2m}, P_v)-group, i.e.

$$\left|P_h^{(2)}\right| = |A_{2m}| \left(\frac{|A_{2m}|}{2m}\right)^{2m}$$

by Proposition 1(1a), then K_h is not a p-group. More precisely

$$|K_h| = |A_{2m-1}|^{2m-1}.$$

We apply now Proposition 38 to a $(4, 6)$-group which is moreover a candidate for having a simple subgroup of index 4.

Example 31.

$$R(2,3) := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_1^{-1}, \\ \\ a_1 b_3 a_2^{-1} b_1, & a_1 b_3^{-1} a_2 b_3, \\ \\ a_1 b_2^{-1} a_2^{-1} b_3^{-1}, & a_2 b_1 a_2^{-1} b_2 \end{array} \right\}.$$

Theorem 31. (1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v = S_6$.

(2) $|K_v| = 12441600000 = 2^{14} \cdot 3^5 \cdot 5^5$.

(3) Γ *is irreducible and not linear over any field.*

(4) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

(5) $Z_\Gamma(b_3) = N_\Gamma(\langle b_3 \rangle) = \langle b_3 \rangle$.

(6) $\mathrm{Aut}(X) \cong \mathbb{Z}_2$.

Proof. (1)

$$\begin{aligned}\rho_v(b_1) &= (1,2),\\ \rho_v(b_2) &= (3,4),\\ \rho_v(b_3) &= (1,2,4,3),\\ \\ \rho_h(a_1) &= (1,2)(3,5,6),\\ \rho_h(a_2) &= (1,4,2,6,5).\end{aligned}$$

(2) GAP ([28]).

(3) By Proposition 38.

(4) GAP ([28]).

(5) This follows from Proposition 8.

(6) GAP ([28]). $\mathrm{Aut}(X)$ is generated by

$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_1^{-1}, a_2^{-1}, b_2, b_1, b_3).$$

□

Conjecture 19. Γ *is non-residually finite such that*

$$\bigcap_{N \overset{f.i.}{\triangleleft} \Gamma} N = \Gamma_0.$$

Question 9. *Is Γ_0 simple?*

Example 32.

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, \\ \\ a_1b_3a_2^{-1}b_1, & a_1b_3^{-1}a_2b_1^{-1}, \\ \\ a_2b_1a_2b_2^{-1}, & a_2b_2a_2b_3 \end{array} \right\}.$$

Theorem 32. (1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $|K_v| = 50000 = 2^4 \cdot 5^5$.

(3) Γ *is irreducible and not linear over any field.*

(4) $[\Gamma, \Gamma] = \Gamma_0$, $\Gamma_0^{ab} \cong \mathbb{Z}_2$, $\Gamma/[\Gamma_0, \Gamma_0] \cong D_4$ *and* $[\Gamma_0, \Gamma_0]$ *is perfect.*

(5) $Z_\Gamma(a_i) = N_\Gamma(\langle a_i \rangle) = \langle a_i \rangle$, *if* $a_i \in \{a_1, a_2\}$.

(6) $\mathrm{Aut}(X) \cong \mathbb{Z}_2$.

Proof. (1)

$$\begin{aligned}\rho_v(b_1) &= (1,3,2),\\ \rho_v(b_2) &= (2,3),\\ \rho_v(b_3) &= (2,4,3),\\ \\ \rho_h(a_1) &= (1,4,5,6,3,2),\\ \rho_h(a_2) &= (1,4,2)(3,6,5).\end{aligned}$$

(2) GAP ([28]).

(3) By Proposition 38.

(4) GAP ([28]).

(5) This follows from Proposition 8.

(6) GAP ([28]). $\mathrm{Aut}(X)$ is generated by

$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_1, a_2^{-1}, b_1^{-1}, b_2^{-1}, b_3^{-1}).$$

$\square$

Conjecture 20. Γ *is non-residually finite such that*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = [\Gamma_0, \Gamma_0].$$

Question 10. *Is* $[\Gamma_0, \Gamma_0]$ *simple?*

Example 33.

$$R(3,3) := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_3^{-1}, & a_1 b_3 a_2^{-1} b_1 \\ a_1 b_3^{-1} a_3^{-1} b_3, & a_1 b_2^{-1} a_2^{-1} b_1^{-1}, & a_2 b_1 a_2^{-1} b_2^{-1} \\ a_2 b_3 a_3^{-1} b_3^{-1}, & a_3 b_1 a_3 b_2, & a_3 b_2^{-1} a_3 b_1^{-1} \end{array} \right\}.$$

Theorem 33. (1) $P_h \cong \mathrm{PSL}_2(5) < S_6$, $P_v \cong \mathrm{PSL}_2(5) < S_6$.

(2) $|K_v| = 100000 = 2^5 \cdot 5^5$.

(3) Γ *is irreducible and not linear over any field.*

(4) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

(5) $Z_\Gamma(b_3) = N_\Gamma(\langle b_3 \rangle) = \langle b_3 \rangle$.

(6) $\mathrm{Aut}(X) \cong \mathbb{Z}_2^2$.

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (1,2)(3,4), \\ \rho_v(b_2) &= (3,4)(5,6), \\ \rho_v(b_3) &= (1,2,3)(4,6,5), \\ \\ \rho_h(a_1) &= (1,5,6,3,2), \\ \rho_h(a_2) &= (1,4,5,6,2), \\ \rho_h(a_3) &= (1,5)(2,6). \end{aligned}$$

(2) GAP ([28]).

(3) By Proposition 38.

(4) GAP ([28]).

(5) This follows from Proposition 8.

(6) GAP ([28]). $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$(a_1, a_2, a_3, b_1, b_2, b_3) \mapsto (a_2, a_1, a_3, b_1^{-1}, b_2^{-1}, b_3^{-1}),$$
$$(a_1, a_2, a_3, b_1, b_2, b_3) \mapsto (a_2^{-1}, a_1^{-1}, a_3^{-1}, b_2, b_1, b_3^{-1}).$$

$\square$

Conjecture 21. Γ *is non-residually finite such that*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = \Gamma_0.$$

Question 11. *Is* Γ_0 *simple?*

4.10 Two very small (irreducible?) examples

Finally, we give two interesting small examples whose properties are not known very well so far.

Example 34.

$$R(2,2) := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_1, \\ & \\ a_2 b_1 a_2 b_2, & a_1 b_2^{-1} a_2 b_1^{-1} \end{array} \right\}.$$

Theorem 34. (1) $P_h = A_4$, $P_v \cong D_4$, *the dihedral group of order* 8.

(2) $\mathrm{QZ}(H_2) \neq 1$.

(3) $\Gamma^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3$, $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z} \times \mathbb{Z}_5^2$, $\Gamma_0^{ab} \cong \mathbb{Z} \times \mathbb{Z}_3$.

(4) Γ *has a quotient* $\mathbb{Z}_2 * \mathbb{Z}_2$.

(5)

$$[\Gamma : \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N] = \infty.$$

(6) Γ *has finite quotients* $\mathrm{PGL}_2(7)$, $\mathrm{PGL}_2(11)$, $\mathrm{PGL}_2(13)$, $\mathrm{PGL}_2(17)$, $\mathrm{PSL}_2(19)$, $\mathrm{PSL}_2(23)$, $\mathrm{PSL}_2(29)$, $\mathrm{PSL}_2(43)$, $\mathrm{PSL}_2(47)$, $\mathrm{PSL}_2(53)$.

Proof. (1) $\rho_v(b_1) = (1,3,2)$, $\rho_v(b_2) = (2,4,3)$, $\rho_h(a_1) = (1,3,4,2)$, $\rho_h(a_2) = (1,3)(2,4)$.

(2) $B := \{b_1^3, b_2^3, b_2^{-3}, b_1^{-3}\} \subset \Lambda_2$, since for each $b \in B$ and $a \in E_h$ we have $\rho_v(b)(a) = a$ and $\rho_h(a)(b) \in B$.

(3) This is an easy computation. Note that a method for trying to show that a group is infinite, is to find a (finite index) subgroup with infinite abelianization.

(4) For example $\Gamma / \langle\langle a_2^2 \rangle\rangle_\Gamma \cong \mathbb{Z}_2 * \mathbb{Z}_2$, since

$$\langle a_1, a_2, b_1, b_2 \mid a_1 b_1 a_1^{-1} b_2^{-1}, a_1 b_2 a_2^{-1} b_1, a_2 b_1 a_2 b_2, a_1 b_2^{-1} a_2 b_1^{-1}, a_2^2 \rangle =$$

$$\langle a_1, a_2, b_1, b_2 \mid b_2 = a_1 b_1 a_1^{-1}, a_1 = b_1^{-1} a_2 b_2^{-1} = a_2, a_1 b_2^{-1} a_2 b_1^{-1}, a_2^2 \rangle =$$
$$\langle a_1, b_1 \mid a_1^2 b_1 a_1^{-2} b_1, a_1 b_1 a_1^2 b_1 a_1^{-1}, a_1^2 b_1^{-2}, a_1^2 \rangle =$$
$$\langle a_1, b_1 \mid a_1^2, b_1^2 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

(5) $|\Gamma/\langle\!\langle a_2^2, [a_1,b_1]^m \rangle\!\rangle_\Gamma| = 4m$, since

$$\langle [a_1,b_1]^m\rangle \overset{m}{\lhd} \langle [a_1,b_1]\rangle \overset{4}{\lhd} \langle a_1, b_1 \mid a_1^2, b_1^2\rangle = \Gamma/\langle\!\langle a_2^2 \rangle\!\rangle_\Gamma.$$

(6) quotpic ([59])

□

Conjecture 22. (1) Γ *is irreducible. We have computed*

	$\lvert P_h^{(k)}\rvert$	$\lvert P_v^{(k)}\rvert$
$k=1$	12	8
$k=2$	324	32
$k=3$	8748	128
$k=4$	236196	1024
$k=5$	6377292	8192
$k=6$	172186884	65536
$k=7$	4649045868	524288
...	...	...

(2) Γ *is residually finite.*

(3) $\mathrm{QZ}(H_1) = 1$.

See Table 12 for the orders of some quotients of Γ. The infinite quotients in this table which do not correspond to elements in Λ_2, are recognized by MAGNUS ([49]).

$\lvert\Gamma/\langle\!\langle w^k\rangle\!\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10
$w=a_1$	6	∞	6	∞	750	∞	146160	∞	2147040	∞
a_2	2	∞	150	∞	2	∞	158928	∞	1026000	∞
b_1	2	∞	∞(QZ)	∞	2	∞(QZ)	2	∞	∞(QZ)	∞
b_2	2	∞	∞(QZ)	∞	2	∞(QZ)	2	∞	∞(QZ)	∞

Table 12: Order of $\Gamma/\langle\!\langle w^k\rangle\!\rangle_\Gamma$, $w \in \{a_1, a_2, b_1, b_2\}$, $k = 1, \dots, 10$, in Example 34

Example 35.

$$R(2,2) := \left\{ \begin{array}{ll} a_1 b_1 a_2^{-1} b_1, & a_1 b_2 a_2 b_2, \\ a_2 b_1 a_2 b_2^{-1}, & a_1 b_1^{-1} a_1 b_2^{-1} \end{array} \right\}.$$

Theorem 35. (1) $P_h = S_4$, $P_v = S_4$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3^4 \times \mathbb{Z}_4$, $\Gamma_0^{ab} \cong \mathbb{Z}_3^2 \times \mathbb{Z}_4$.

(3) Γ *has finite quotients* $\mathrm{PGL}_2(9)$, $\mathrm{PSL}_2(27)$.

Proof. (1) $\rho_v(b_1) = (1,4,3,2)$, $\rho_v(b_2) = (1,4,2,3)$, $\rho_h(a_1) = (1,3,2,4)$, $\rho_h(a_2) = (1,4,3,2)$.

(2) Easy computation.

(3) quotpic ([59])

□

Conjecture 23. (1) Γ *is irreducible. We have computed*

	$\lvert P_h^{(k)}\rvert$	$\lvert P_v^{(k)}\rvert$
$k=1$	24	24
$k=2$	648	648
$k=3$	17496	17496
$k=4$	472392	472392
$k=5$	12754584	12754584
$k=6$	344373768	344373768
$k=7$	9298091736	9298091736
...	...	...

(2) Γ *is residually finite.*

(3) $\mathrm{QZ}(H_1) = \mathrm{QZ}(H_2) = 1$.

(4)

$$[\Gamma : \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N] = \infty.$$

(5) *Any non-trivial normal subgroup of* Γ *has finite index.*

See Table 13 for the orders of some quotients of Γ.

$\lvert\Gamma/\langle\langle w^k\rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8
$w=a_1$	6	24	162	48	4320	17496	117936	17280
$w=a_2$	6	24	162	48	4320	17496	117936	17280
$w=b_1$	6	24	162	48	4320	17496	117936	17280
$w=b_2$	6	24	162	48	4320	17496	117936	17280

Table 13: Order of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$, $w \in \{a_1, a_2, b_1, b_2\}$, $k = 1, \dots, 8$, in Example 35

4.11 Maximal $P_h^{(2)}$, $P_v^{(2)}$

Example 36.

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_3^{-1}a_3^{-1}b_3, & a_2b_1a_2^{-1}b_2^{-1}, & a_2b_2a_3^{-1}b_1, \\ a_2b_3a_3b_1^{-1}, & a_2b_2^{-1}a_3b_3^{-1}, & a_3b_1a_3b_2 \end{array} \right\}.$$

Theorem 36. (1) $P_h = S_6$, $P_v = S_6$, $\left|P_h^{(2)}\right| = \left|P_v^{(2)}\right| = 720 \cdot 120^6$.

(2) $\left|P_h^{(3)}\right|$ *and* $\left|P_v^{(3)}\right|$ *are not maximal, i.e. smaller than the corresponding groups of the local action of* $\mathrm{Aut}(\mathcal{T}_6)$ *on* $\mathcal{T}_6$.

Proof. (1) We have computed it with GAP ([28]), using the programs of Appendix D.4.

(2) $\left|P_h^{(3)}\right| = \left|P_v^{(3)}\right| = 720 \cdot 120^6 \cdot 120^{30}/64$ again using GAP ([28]).

□

4.12 Local groups of Γ_0

Our definition of $P_h^{(k)}$, $P_v^{(k)}$ for $(2m, 2n)$-complexes fits in the definition of local groups for more general square complexes given in [16, Chapter 1]. In this more general context, local groups are defined for each vertex of the complex. The next three examples have the same P_h, P_v, $P_h^{(2)}$, $P_v^{(2)}$, but different local groups for X_0, denoted by $P_h(X_0)$, $P_v(X_0)$. These groups do not depend on the four vertices of X_0 in the following examples.

Example 37.

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2, \\ a_1b_3^{-1}a_3^{-1}b_2^{-1}, & a_1b_2^{-1}a_3^{-1}b_3, & a_2b_1a_3b_1, \\ a_2b_2a_2b_1^{-1}, & a_2b_3a_3^{-1}b_3^{-1}, & a_3b_2^{-1}a_3b_1^{-1} \end{array} \right\}.$$

Theorem 37. *(1) $P_h \cong P_v \cong \mathrm{PGL}_2(5)$ and $|P_h^{(2)}| = |P_v^{(2)}| = 15000$.*

(2) $P_h(X_0) \cong \mathrm{PGL}_2(5)$, $P_v(X_0) \cong \mathrm{PGL}_2(5)$.

Proof. GAP □

Example 38.

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_3^{-1}b_2, \\ a_1b_3^{-1}a_3b_2^{-1}, & a_1b_2^{-1}a_2b_3, & a_2b_1a_3^{-1}b_1^{-1}, \\ a_2b_2a_3b_1, & a_2b_3^{-1}a_3^{-1}b_3^{-1}, & a_2b_1^{-1}a_3b_2 \end{array} \right\}.$$

Theorem 38. *(1) $P_h \cong P_v \cong \mathrm{PGL}_2(5)$ and $|P_h^{(2)}| = |P_v^{(2)}| = 15000$.*

(2) $P_h(X_0) \cong \mathrm{PSL}_2(5)$, $P_v(X_0) \cong \mathrm{PGL}_2(5)$.

Proof. GAP □

Example 39.

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_3^{-1}b_2, \\ a_1b_3^{-1}a_3b_2^{-1}, & a_1b_2^{-1}a_2b_3, & a_2b_1a_3^{-1}b_3^{-1}, \\ a_2b_2a_3b_1, & a_2b_3^{-1}a_3^{-1}b_1^{-1}, & a_2b_1^{-1}a_3b_2 \end{array} \right\}.$$

Theorem 39. *(1) $P_h \cong P_v \cong \mathrm{PGL}_2(5)$ and $|P_h^{(2)}| = |P_v^{(2)}| = 15000$.*

(2) $P_h(X_0) \cong \mathrm{PSL}_2(5)$, $P_v(X_0) \cong \mathrm{PSL}_2(5)$.

Proof. GAP □

5 Quaternion lattices in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$

5.1 Some notations and preliminaries

We first define quaternions over a ring (see e.g. [22, Chapter 2.4]).

Definition. Let R be a commutative ring with unit. Then the *Hamilton quaternion algebra* over R, denoted by $\mathbb{H}(R)$, is the associative unital algebra defined as follows:

- $\mathbb{H}(R) = \{x = x_0 + x_1 i + x_2 j + x_3 k \mid x_0, x_1, x_2, x_3 \in R\}$ is the free R-module with basis 1, i, j, k.
- $1 = 1 + 0i + 0j + 0k$ is the multiplicative unit.
- $i^2 = j^2 = k^2 = -1$.
- $ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$.

This gives the multiplication rule

$$\begin{aligned}(x_0 + x_1 i + x_2 j + x_3 k)(y_0 + y_1 i + y_2 j + y_3 k) = {} & x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \\ & + (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2) i \\ & + (x_0 y_2 - x_1 y_3 + x_2 y_0 + x_3 y_1) j \\ & + (x_0 y_3 + x_1 y_2 - x_2 y_1 + x_3 y_0) k.\end{aligned}$$

For a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(R)$, let $\bar{x} := x_0 - x_1 i - x_2 j - x_3 k$ be its conjugate, $|x|^2 := x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2$ its norm, and $\mathrm{Re}(x) := x_0$ its "R-part". Note that $|xy|^2 = |x|^2 |y|^2$. We divide quaternions $x \in \mathbb{H}(\mathbb{Z})$ with *odd* norm $|x|^2$ into eight classes (and say that these quaternions have type o_0, o_1, o_2, o_3, e_0, e_1, e_2 or e_3) according to Table 14. We say

x	x_0	x_1	x_2	x_3
type o_0	odd	even	even	even
o_1	even	odd	even	even
o_2	even	even	odd	even
o_3	even	even	even	odd
e_0	even	odd	odd	odd
e_1	odd	even	odd	odd
e_2	odd	odd	even	odd
e_3	odd	odd	odd	even

Table 14: Types of integer quaternions x with odd norm $|x|^2$.

that x has type o if it has type o_0, o_1, o_2 or o_3. Note that x has type o if and only if $|x|^2 \equiv 1 \pmod 4$.

If R is a ring with unit (denoted by 1), let $U(R)$ be the group of (left and right) invertible elements in R, i.e. elements $x \in R$ such that there are $y_1, y_2 \in R$ satisfying $y_1 x = x y_2 = 1$. Note that then $y_1 = y_2$. This element is uniquely determined by $x \in U(R)$ and is usually written as x^{-1}.

Lemma 39. *Let R be a commutative ring with unit. Then*

$$U(\mathbb{H}(R)) = \{x \in \mathbb{H}(R) : |x|^2 \in U(R)\}.$$

Proof. "$\supseteq$" Take $x^{-1} = (|x|^2)^{-1}\bar{x}$.
"$\subseteq$" Let $x \in U(\mathbb{H}(R))$ and $y := x^{-1}$, then $1 = |xy|^2 = |x|^2|y|^2 = |y|^2|x|^2$, i.e. $|x|^2 \in U(R)$. $\square$

Lemma 40. *Let R be a subring (with unit) of $\mathbb{C}$, then*

(1) $\{x \in \mathbb{H}(R) : xy = yx, \forall y \in \mathbb{H}(R)\} = \{x \in \mathbb{H}(R) : x = \bar{x}\} = \{x \in \mathbb{H}(R) : x = \mathrm{Re}(x)\}$.

(2) $ZU(\mathbb{H}(R)) = \{x \in U(\mathbb{H}(R)) : x = \mathrm{Re}(x)\} = U(\mathbb{H}(R)) \cap ZU(\mathbb{H}(\mathbb{C}))$.

Proof. (1) We prove the first equality, the second one is obvious. Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(R)$. If $x = \mathrm{Re}(x)$ (i.e. $x_1 = x_2 = x_3 = 0$), then clearly $xy = yx$ for all $y \in \mathbb{H}(R)$. For the other direction, suppose that $xy = yx$ for all $y \in \mathbb{H}(R)$. Taking $y = i$, the condition $xi = ix$ is equivalent to

$$-x_1 + x_0 i + x_3 j - x_2 k = -x_1 + x_0 i - x_3 j + x_2 k$$

and it follows $x_2 = x_3 = 0$. Taking $y = j$, we conclude in the same way $x_1 = x_3 = 0$, thus $x = x_0$.

(2) We can use the same proof as in (1), since $i(-i) = j(-j) = 1$, i.e. $i, j \in U(\mathbb{H}(R))$.

$\square$

Remark. The case $R = \mathbb{Z}_2$ is different:

$$ZU(\mathbb{H}(\mathbb{Z}_2)) = U(\mathbb{H}(\mathbb{Z}_2)) \neq \{x \in U(\mathbb{H}(\mathbb{Z}_2)) : x = \mathrm{Re}(x)\} = \{1\}.$$

Lemma 41. *Let R be a commutative ring with unit. Let $x = x_0 + x_1 i + x_2 j + x_3 k$, $y = y_0 + y_1 i + y_2 j + y_3 k$, $z = z_0 + z_1 i + z_2 j + z_3 k \in \mathbb{H}(R)$. Then*

(1) *$xy = yx$ if and only if $2(x_2 y_3 - x_3 y_2) = 0$ and $2(x_3 y_1 - x_1 y_3) = 0$ and $2(x_1 y_2 - x_2 y_1) = 0$.*

(2) *$xy = -yx$ if and only if $2(x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3) = 0$ and $2(x_0 y_1 + x_1 y_0) = 0$ and $2(x_0 y_2 + x_2 y_0) = 0$ and $2(x_0 y_3 + x_3 y_0) = 0$.*

(3) *Suppose that R is a subring of $\mathbb{R}$ with unit, $x_0 \neq 0$ and $xy = -yx$. Then $y = 0$.*

(4) *Let R be a subring of $\mathbb{R}$ with unit, $x \neq x_0$, $xy = yx$ and $xz = zx$. Then $yz = zy$, in particular $U(\mathbb{H}(\mathbb{R}))$ is commutative transitive on non-central elements.*

Proof. (1) and (2) are elementary computations using the multiplication rule for quaternions.

(3) Using (2), we have $x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 = 0$ and

$$y_1 = \frac{-x_1 y_0}{x_0}, \quad y_2 = \frac{-x_2 y_0}{x_0}, \quad y_3 = \frac{-x_3 y_0}{x_0}.$$

It follows

$$x_0 y_0 + \frac{x_1^2 y_0}{x_0} + \frac{x_2^2 y_0}{x_0} + \frac{x_3^2 y_0}{x_0} = 0,$$

i.e. $y_0 |x|^2 = 0$. Since $|x|^2 \geq x_0^2 > 0$, we conclude $y_0 = 0$ which implies $y_1 = 0$, $y_2 = 0$ and $y_3 = 0$, i.e. $y = 0$.

(4) By (1), we have to prove $y_2 z_3 = y_3 z_2$, $y_3 z_1 = y_1 z_3$ and $y_1 z_2 = y_2 z_1$. We only prove here $y_1 z_2 = y_2 z_1$, the other two computations are completely analogous: If $x_2 = 0$, then $x_1 y_2 = x_2 y_1 = 0$ and $x_3 y_2 = x_2 y_3 = 0$. This implies $y_2 = 0$ (otherwise $x_1 = x_3 = 0$ and $x = x_0$). Moreover, we have $x_1 z_2 = x_2 z_1 = 0$ and $x_3 z_2 = x_2 z_3 = 0$, which implies $z_2 = 0$. So, we conclude that $y_1 z_2 = 0 = y_2 z_1$. Assume now that $x_2 \neq 0$, then $y_1 z_2 = \frac{x_1}{x_2} y_2 z_2 = y_2 z_1$, using $x_2 y_1 = x_1 y_2$ and $x_2 z_1 = x_1 z_2$.

$\square$

Throughout this chapter, let p and l be two distinct odd primes. Then

$$\mathbb{Z}[1/p, 1/l] := \{0\} \cup \{tp^r l^s : r, s, t \in \mathbb{Z}; t \neq 0;\ t, p \text{ and } t, l \text{ are relatively prime}\}$$

is a subring of $\mathbb{Q}$, containing $\mathbb{Z}$.

Let $\left(\frac{p}{l}\right)$ be the *Legendre symbol*. This means that $\left(\frac{p}{l}\right) := 1$, if p is a quadratic residue modulo l, i.e. if the equation $x^2 \equiv p \pmod{l}$ has an integer solution, and $\left(\frac{p}{l}\right) := -1$, otherwise. See Table 15 for some examples. The definition of the Legendre symbol can be generalized to non-prime numbers, but we do not need it here.

Let K be a field, $K^\times = K \setminus \{0\} = U(K)$ the group of invertible elements and $\mathrm{GL}_2(K)$ the group of invertible (2×2)-matrices with coefficients in K. We denote by $\mathrm{PGL}_2(K)$ the quotient group

$$\mathrm{PGL}_2(K) = \mathrm{GL}_2(K)/\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in K^\times\} = \mathrm{GL}_2(K)/Z\mathrm{GL}_2(K).$$

If A is a matrix in $\mathrm{GL}_2(K)$, we write

$$[A] := A\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in K^\times\} \in \mathrm{PGL}_2(K)$$

for the image of A under the quotient map $\mathrm{GL}_2(K) \to \mathrm{PGL}_2(K)$.

We denote by $\mathrm{SL}_2(K)$ the kernel of the determinant map $\det : \mathrm{GL}_2(K) \to K^\times$ and by $\mathrm{PSL}_2(K)$ the quotient group

$$\mathrm{PSL}_2(K) = \mathrm{SL}_2(K)/\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} : \epsilon = \pm 1\} = \mathrm{SL}_2(K)/Z\mathrm{SL}_2(K).$$

$\mathrm{PSL}_2(K)$ can be seen as a subgroup of $\mathrm{PGL}_2(K)$ via the injective homomorphism

$$\mathrm{PSL}_2(K) \to \mathrm{PGL}_2(K)$$
$$A\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} : \epsilon = \pm 1\} \mapsto [A],$$

where $A \in \mathrm{SL}_2(K)$.

For q prime, we write $\mathrm{GL}_2(q)$, $\mathrm{PGL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ instead of $\mathrm{GL}_2(\mathbb{Z}_q)$, $\mathrm{PGL}_2(\mathbb{Z}_q)$, $\mathrm{SL}_2(\mathbb{Z}_q)$, $\mathrm{PSL}_2(\mathbb{Z}_q)$. We want to emphasize that $\mathbb{Z}_q$ stands for the finite ring (field) $\mathbb{Z}/q\mathbb{Z}$ (as in all other chapters) and *not* for the q-adic integers.

Lemma 42. *Let K be a field and $B \in \mathrm{GL}_2(K)$. Then $[B] \in \mathrm{PSL}_2(K) < \mathrm{PGL}_2(K)$ if and only if $\det B \in (K^\times)^2 := \{k^2 : k \in K^\times\}$.*

Proof. Note that $[B] \in \mathrm{PSL}_2(K)$ if and only if there is a matrix $A \in \mathrm{SL}_2(K)$ such that $[A] = [B] \in \mathrm{PGL}_2(K)$, i.e. if and only if there is a matrix $A \in \mathrm{SL}_2(K)$ and an element $\lambda \in K^\times$ such that

$$B^{-1}A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

To prove the statement of the Lemma, we first assume that $[B] \in \mathrm{PSL}_2(K)$. Then (with A and λ as above)

$$\det B = \det A \cdot \lambda^{-2} = \lambda^{-2} \in (K^\times)^2.$$

To show the other direction, assume that $\det B = k^2$ for some $k \in K^\times$. If we choose

$$A := B \begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix},$$

then $A \in \mathrm{SL}_2(K)$, since $\det A = k^2 \cdot k^{-2} = 1$, and we have

$$B^{-1}A = \begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix}.$$

□

Lemma 43. *Let p, l be two distinct odd primes. Then $p + l\mathbb{Z} \in (\mathbb{Z}_l^\times)^2$ if and only if $\left(\frac{p}{l}\right) = 1$.*

Proof.

$$\begin{aligned} p + l\mathbb{Z} \in (\mathbb{Z}_l^\times)^2 &\Leftrightarrow \exists\, x + l\mathbb{Z} \in \mathbb{Z}_l^\times \text{ such that } (x + l\mathbb{Z})^2 = p + l\mathbb{Z} \\ &\Leftrightarrow \exists\, x \in \{1, \ldots, l-1\} \text{ such that } x^2 + l\mathbb{Z} = p + l\mathbb{Z} \\ &\Leftrightarrow \exists\, x \in \{1, \ldots, l-1\} \text{ such that } x^2 \equiv p \pmod{l} \\ &\Leftrightarrow \exists\, x \in \mathbb{Z} \text{ such that } x^2 \equiv p \pmod{l} \\ &\Leftrightarrow \left(\frac{p}{l}\right) = 1. \end{aligned}$$

□

The next lemma gives some old results which are well-known in number theory.

Lemma 44. *Let p be an odd prime and s an odd natural number.*

(1) *(Fermat, Euler) p is a sum of 2 squares if and only if $p \equiv 1 \pmod 4$.*

(2) *(Gauss) Assume that $p \equiv 3 \pmod 4$. Then p is a sum of 3 squares if and only if $p \equiv 3 \pmod 8$. More generally, s is a sum of 3 squares if and only if s is not $\equiv 7 \pmod 8$.*

(3) *(Jacobi) p has exactly $8(p+1)$ representations as a sum of 4 squares $p = x_0^2 + x_1^2 + x_2^2 + x_3^2$; $x_0, x_1, x_2, x_3 \in \mathbb{Z}$. For each such representation, three integers in $\{x_0, x_1, x_2, x_3\}$ are even, if $p \equiv 1 \pmod 4$, and three integers are odd, if $p \equiv 3 \pmod 4$. It follows that*

$$|\{x \in \mathbb{H}(\mathbb{Z}) : |x|^2 = p \equiv 1 \pmod 4,\ x \text{ has type } o\}| = 8(p+1),$$

$$|\{x \in \mathbb{H}(\mathbb{Z}) : |x|^2 = p \equiv 1 \pmod 4,\ x \text{ has type } o_0\}| = 2(p+1)$$

and

$$|\{x \in \mathbb{H}(\mathbb{Z}) : |x|^2 = p \equiv 1 \pmod 4,\ x \text{ has type } o_0,\ \mathrm{Re}(x) > 0\}| = p+1.$$

Let p be an odd prime. The following lemma applies for example to the finite field $\mathbb{Z}_p$, the field of p-adic numbers $\mathbb{Q}_p$ and algebraically closed fields like $\mathbb{C}$, but not to $\mathbb{Z}_2$ and subfields of $\mathbb{R}$.

Lemma 45. *(see [22, Propostion 2.4.2]) Let K be a field, not of characteristic 2, and assume that there exist $c, d \in K$ such that $c^2 + d^2 + 1 = 0$. Then $\mathbb{H}(K)$ is isomorphic to the algebra $M_2(K)$ of (2×2)-matrices over K. An isomorphism of algebras is given by*

$$\begin{aligned} \mathbb{H}(K) &\to M_2(K) \\ x_0 + x_1 i + x_2 j + x_3 k &\mapsto \begin{pmatrix} x_0 + x_1 c + x_3 d & -x_1 d + x_2 + x_3 c \\ -x_1 d - x_2 + x_3 c & x_0 - x_1 c - x_3 d \end{pmatrix}. \end{aligned}$$

In particular, if $c^2 + 1 = 0$ in K (i.e. if we can choose $d = 0$), then the isomorphism above is given by

$$\begin{aligned} \mathbb{H}(K) &\to M_2(K) \\ x_0 + x_1 i + x_2 j + x_3 k &\mapsto \begin{pmatrix} x_0 + x_1 c & x_2 + x_3 c \\ -x_2 + x_3 c & x_0 - x_1 c \end{pmatrix}. \end{aligned}$$

Note that

$$\det\begin{pmatrix} x_0 + x_1c + x_3d & -x_1d + x_2 + x_3c \\ -x_1d - x_2 + x_3c & x_0 - x_1c - x_3d \end{pmatrix} = x_0^2 - x_1^2(c^2+d^2) + x_2^2 - x_3^2(c^2+d^2) = |x|^2.$$

$\left(\frac{p}{l}\right)$	$l=3$	5	7	11	13	17	19	23	29	31	37	41	43	47
$p=3$		−1	−1	1	1	−1	−1	1	−1	−1	1	−1	−1	1
5	−1		−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1
7	1	−1		−1	−1	−1	1	−1	1	1	1	−1	−1	1
11	−1	1	1		−1	−1	1	−1	−1	−1	1	−1	1	−1
13	1	−1	−1	−1		1	−1	1	1	−1	−1	−1	1	−1
17	−1	−1	−1	−1	1		1	−1	−1	−1	−1	−1	1	1
19	1	1	−1	−1	−1	1		−1	−1	1	−1	−1	−1	−1
23	−1	−1	1	1	1	−1	1		1	−1	−1	1	1	−1
29	−1	1	1	−1	1	−1	−1	1		−1	−1	−1	−1	−1
31	1	1	−1	1	−1	−1	−1	1	−1		−1	1	1	−1
37	1	−1	1	1	−1	−1	−1	−1	−1	−1		1	−1	1
41	−1	1	−1	−1	−1	−1	−1	1	−1	1	1		1	−1
43	1	−1	1	−1	1	1	1	−1	−1	−1	−1	1		−1
47	−1	−1	−1	1	−1	1	1	1	−1	1	1	−1	1	

Table 15: Legendre symbol $\left(\frac{p}{l}\right)$ for small distinct odd primes p, l.

5.2 $p, l \equiv 1 \pmod 4$

The following construction of the group $\Gamma_{p,l}$ is taken from [53], see also [52], [16] and [39]. Let $p, l \equiv 1 \pmod 4$ be two distinct primes. We define the map

$$\psi : \mathbb{H}(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$
$$x = x_0 + x_1i + x_2j + x_3k \mapsto \left(\left[\begin{pmatrix} x_0 + x_1i_p & x_2 + x_3i_p \\ -x_2 + x_3i_p & x_0 - x_1i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1i_l & x_2 + x_3i_l \\ -x_2 + x_3i_l & x_0 - x_1i_l \end{pmatrix} \right] \right),$$

where $i_p \in \mathbb{Q}_p$, $i_l \in \mathbb{Q}_l$ satisfy $i_p^2 + 1 = 0$, $i_l^2 + 1 = 0$. The assumption $p, l \equiv 1 \pmod 4$ guarantees the existence of such i_p, i_l. Note that ψ is not injective, but (for $x, y \in \mathbb{H}(\mathbb{Z})$) we have $\psi(x) = \psi(y)$ if and only if $y = \lambda x$ for some $\lambda \in \mathbb{Q}^\times$. Moreover,

$$\begin{pmatrix} x_0 + x_1i_p & x_2 + x_3i_p \\ -x_2 + x_3i_p & x_0 - x_1i_p \end{pmatrix} \begin{pmatrix} y_0 + y_1i_p & y_2 + y_3i_p \\ -y_2 + y_3i_p & y_0 - y_1i_p \end{pmatrix} = \begin{pmatrix} z_0 + z_1i_p & z_2 + z_3i_p \\ -z_2 + z_3i_p & z_0 - z_1i_p \end{pmatrix},$$

where z_0, z_1, z_2, z_3 are determined by

$$z_0 + z_1i + z_2j + z_3k = (x_0 + x_1i + x_2j + x_3k)(y_0 + y_1i + y_2j + y_3k),$$

in particular $\psi(xy) = \psi(x)\psi(y)$ and

$$\begin{aligned} \ker(\psi) &:= \{x \in \mathbb{H}(\mathbb{Z}) : \psi(x) = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right)\} \\ &= \{x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\} : x = \mathrm{Re}(x)\} = \mathbb{H}(\mathbb{Z}) \cap ZU(\mathbb{H}(\mathbb{Q})). \end{aligned}$$

Finally, let

$$\Gamma_{p,l} := \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r l^s;\ r,s \in \mathbb{N}_0\}$$
$$= \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \mathrm{Re}(x) > 0,\ |x|^2 = p^r l^s;\ r,s \in \mathbb{N}_0\}.$$

Then $\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l) < \mathrm{Aut}(\mathcal{T}_{p+1}) \times \mathrm{Aut}(\mathcal{T}_{l+1})$ is a $(p+1, l+1)$-group. For a proof, see [53, Section 3], and cf. Corollary 47. See [66] or [43, Chapter 5.3] for the description of the tree (Bruhat-Tits building) $\mathcal{T}_{p+1}$ corresponding to $\mathrm{PGL}_2(\mathbb{Q}_p)$ and its action on $\mathcal{T}_{p+1}$. We can identify the set of standard generators (and their inverses) of the $(p+1, l+1)$-group

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2)\rangle$$

as follows:

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \mathrm{Re}(x) > 0,\ |x|^2 = p\} \subset \Gamma_{p,l}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \mathrm{Re}(x) > 0,\ |x|^2 = l\} \subset \Gamma_{p,l},$$

where we have

$$(\psi(x_0 + x_1 i + x_2 j + x_3 k))^{-1} = \psi(x_0 - x_1 i - x_2 j - x_3 k),$$

i.e. $\psi(x)^{-1} = \psi(\bar{x})$. In particular, we have by Corollary 7(1) two non-abelian free subgroups in $\Gamma_{p,l}$:

$$F_{\frac{p+1}{2}} \cong \langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r,\ r \in \mathbb{N}_0\} < \Gamma_{p,l}$$

and

$$F_{\frac{l+1}{2}} \cong \langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = l^s,\ s \in \mathbb{N}_0\} < \Gamma_{p,l}.$$

We can see $\mathrm{PSL}_2(\mathbb{Q}_p)$ as a subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$ of index $4 = |\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2|$. With the identification from above, we have $\{a_1, \ldots, a_{\frac{p+1}{2}}\} \subset \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l) < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$ if and only if $\left(\frac{p}{l}\right) = 1$, and $\{b_1, \ldots, b_{\frac{l+1}{2}}\} \subset \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l) < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$ if and only if $\left(\frac{l}{p}\right) = 1$. This follows from Lemma 42 (and Hensel's Lemma), see also [15, p.134]. Note that our assumption $p, l \equiv 1 \pmod 4$ implies $\left(\frac{p}{l}\right) = \left(\frac{l}{p}\right)$ by the famous law of quadratic reciprocity, see e.g. [22, Theorem 2.2.2 iii)]. For $\Gamma = \Gamma_{p,l}$, we observe that the index 4 subgroup Γ_0 is characterized as

$$\Gamma_0 = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r} l^{2s};\ r,s \in \mathbb{N}_0\} < \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l).$$

The fact that $\Gamma_{p,l}$ is a $(p+1, l+1)$-group is mainly based on a factorization property for integral quaternions, first proved by Leonard E. Dickson ([23]):

Proposition 46. *(Dickson [23, Theorem 8]) Let $x \in \mathbb{H}(\mathbb{Z})$ be of odd norm and let $|x|^2 = p_1 \ldots p_t$ be the prime decomposition of $|x|^2$, where the factors p_i are arranged in an arbitrary but definite order. Then x can be decomposed as $x = x^{(1)} \ldots x^{(t)}$ such that $x^{(i)} \in \mathbb{H}(\mathbb{Z})$ and $|x^{(i)}|^2 = p_i$, $i = 1, \ldots, t$. This decomposition is uniquely determined up to multiplication of the factors $x^{(i)}$ with a unit $\pm 1, \pm i, \pm j, \pm k$ (and up to the decomposition of prime numbers dividing x, if such numbers exist).*

Corollary 47. *Let $p, l \equiv 1 \pmod 4$ be distinct odd primes. Recall that*

$$\Gamma_{p,l} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \textit{ has type } o_0,\ \mathrm{Re}(x) > 0,\ |x|^2 = p^r l^s;\ r,s \in \mathbb{N}_0\}.$$

Let

$$E_h := \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \mathrm{Re}(x) > 0,\ |x|^2 = p\} \subset \Gamma_{p,l}$$

and

$$E_v := \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \mathrm{Re}(x) > 0,\ |x|^2 = l\} \subset \Gamma_{p,l}.$$

If $\psi(x) \in E_h$ then also $\psi(\bar{x}) = \psi(x)^{-1} \in E_h$. By Lemma 44(3), the set E_h has $p+1$ elements. For these reasons, we write

$$E_h = \{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1}$$

and similarly

$$E_v = \{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1}.$$

(1) *Let $x \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|x|^2 = pl$. Then there are $y, \tilde{y}, z, \tilde{z} \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|y|^2 = |\tilde{y}|^2 = p$, $|z|^2 = |\tilde{z}|^2 = l$ and $yz = x = \tilde{z}\tilde{y}$. The quaternions $y, \tilde{y}, z, \tilde{z}$ are uniquely determined by x up to sign.*

(2) *Let $a \in E_h$, $b \in E_v$. Then there are unique $\tilde{a} \in E_h$, $\tilde{b} \in E_v$ such that $ab = \tilde{b}\tilde{a}$.*

(3) *The group $\Gamma_{p,l}$ is generated by $\{a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}\}$.*

(4) *Let*

$$\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\} := \{x \in \mathbb{H}(\mathbb{Z}) : x \text{ has type } o_0, \mathrm{Re}(x) > 0, |x|^2 = p\}.$$

Let $x \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|x|^2 = p^r$ for some $r \in \mathbb{N}_0$. Then there is a unique representation

$$x = \pm p^{r_1} w_{r_2}(\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}),$$

where $r_1, r_2 \in \mathbb{N}_0$, $2r_1 + r_2 = r$ and $w_{r_2}(\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1})$ denotes a reduced word of length r_2 in $\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}$ (reduced means here that there are no subwords of the form $\alpha_i\overline{\alpha_i}$ or $\overline{\alpha_i}\alpha_i$).

(5) $\langle a_1, \dots, a_{\frac{p+1}{2}}\rangle \cong F_{\frac{p+1}{2}}$ *and* $\langle b_1, \dots, b_{\frac{l+1}{2}}\rangle \cong F_{\frac{l+1}{2}}$.

Proof. We define a map $u : \{x \in \mathbb{H}(\mathbb{Z}) : x \text{ has type } o\} \to \{1, i, j, k\}$ by

$$u(x) := \begin{cases} 1, & x \text{ has type } o_0, \\ i, & x \text{ has type } o_1, \\ j, & x \text{ has type } o_2, \\ k, & x \text{ has type } o_3. \end{cases}$$

Note that $xu(x)$ always has type o_0.

(1) By Proposition 46 there are $\hat{y}, \hat{z} \in \mathbb{H}(\mathbb{Z})$ such that $|\hat{y}|^2 = p$, $|\hat{z}|^2 = l$ and $x = \hat{y}\hat{z}$. Since $p, l \equiv 1 \pmod 4$, the quaternions $\hat{y}$ and $\hat{z}$ have type o. They have both the same type since $x = \hat{y}\hat{z}$ has type o_0. If $\hat{y}$ and $\hat{z}$ have type o_0, we take $y := \hat{y}$, $z := \hat{z}$ and are done. If $\hat{y}$ and $\hat{z}$ have type o_1, o_2 or o_3, we take $y := -\hat{y}u(\hat{y})$, $z := u(\hat{z})\hat{z}$ and get $yz = -\hat{y}u(\hat{y})u(\hat{z})\hat{z} = -\hat{y}(-1)\hat{z} = x$. The uniqueness up to sign of y and z follows from the uniqueness statement in Proposition 46. Analogously, one proves $x = \tilde{z}\tilde{y}$.

(2) a and b uniquely determine $y, z \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $\mathrm{Re}(y) > 0$, $\mathrm{Re}(z) > 0$, $|y|^2 = p$, $|z|^2 = l$ and $\psi(y) = a$, $\psi(z) = b$. It follows that yz has type o_0 and $|yz|^2 = pl$. By (1), there are $\tilde{y}, \tilde{z} \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|\tilde{y}|^2 = p$, $|\tilde{z}|^2 = l$ and $yz = \tilde{z}\tilde{y}$. Moreover, $\tilde{y}, \tilde{z}$ are uniquely determined up to sign. In particular, there are unique $\tilde{y}, \tilde{z} \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|\tilde{y}|^2 = p$, $|\tilde{z}|^2 = l$, $\mathrm{Re}(\tilde{y}) > 0$, $\mathrm{Re}(\tilde{z}) > 0$ and $\tilde{z}\tilde{y} \in \{yz, -yz\}$. Now take $\tilde{b} := \psi(\tilde{z}) \in E_v$ and $\tilde{a} := \psi(\tilde{y}) \in E_h$. Since $ab = \psi(yz) = \psi(-yz) = \psi(\tilde{z}\tilde{y}) = \tilde{b}\tilde{a}$, the claim follows.

(3) Fix any element $x \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $\mathrm{Re}(x) > 0$ and $|x|^2 \in \{p^r l^s : r, s \in \mathbb{N}_0\}$. We may assume that $r > 0$ or $s > 0$. By Proposition 46, there is a decomposition

$$x = y^{(1)} \dots y^{(r)} z^{(1)} \dots z^{(s)}$$

such that $y^{(1)}, \dots, y^{(r)} \in \mathbb{H}(\mathbb{Z})$ have norm p and $z^{(1)}, \dots, z^{(s)} \in \mathbb{H}(\mathbb{Z})$ have norm l. Note that $y^{(1)}, \dots, y^{(r)}, z^{(1)}, \dots, z^{(s)}$ have type o, since $p, l \equiv 1 \pmod 4$. Our goal is to have a decomposition

$$x = \hat{y}^{(1)} \dots \hat{y}^{(r)} \hat{z}^{(1)} \dots \hat{z}^{(s)}$$

such that $\hat{y}^{(1)}, \dots, \hat{y}^{(r)}$ and $\hat{z}^{(1)}, \dots, \hat{z}^{(s)}$ have norm p and l, respectively, and are moreover of type o_0. To achieve this, we define the following algorithm:

$$\begin{array}{lll}
\tilde{y}^{(1)} & := y^{(1)}, & \\
\tilde{y}^{(i)} & := u(\tilde{y}^{(i-1)}) y^{(i)}, & i = 2, \dots, r, \\
\hat{y}^{(i)} & := \tilde{y}^{(i)} u(\tilde{y}^{(i)}), & i = 1, \dots, r-1, \\
\hat{y}^{(r)} & := \tilde{y}^{(r)} u(\tilde{y}^{(r)}), & \text{if } s \geq 1, \\
\hat{y}^{(r)} & := \tilde{y}^{(r)} & \text{if } s = 0, \\
\tilde{z}^{(1)} & := u(\tilde{y}^{(r)}) z^{(1)}, & \text{if } r \geq 1, \\
\tilde{z}^{(1)} & := z^{(1)}, & \text{if } r = 0, \\
\tilde{z}^{(j)} & := u(\tilde{z}^{(j-1)}) z^{(j)}, & j = 2, \dots, s, \\
\hat{z}^{(j)} & := \tilde{z}^{(j)} u(\tilde{z}^{(j)}), & j = 1, \dots, s-1, \\
\hat{z}^{(s)} & := \tilde{z}^{(s)}. &
\end{array}$$

By construction, $|\hat{y}^{(i)}|^2 = |\tilde{y}^{(i)}|^2 = |y^{(i)}|^2 = p$, $i = 1, \dots, r$, $|\hat{z}^{(j)}|^2 = |\tilde{z}^{(j)}|^2 = |z^{(j)}|^2 = l$, $j = 1, \dots, s$, and $\hat{y}^{(1)}, \dots, \hat{y}^{(r-1)}, \hat{z}^{(1)}, \dots, \hat{z}^{(s-1)}$ have type o_0. Moreover,

$$\begin{aligned}
x &= y^{(1)} y^{(2)} y^{(3)} \dots y^{(r)} z^{(1)} \dots z^{(s)} \\
&= \pm \underbrace{y^{(1)} u(y^{(1)})}_{=\hat{y}^{(1)}} \underbrace{u(y^{(1)}) y^{(2)}}_{=\tilde{y}^{(2)}} y^{(3)} \dots y^{(r)} z^{(1)} \dots z^{(s)} \\
&= \pm \hat{y}^{(1)} \underbrace{\tilde{y}^{(2)} u(\tilde{y}^{(2)})}_{=\hat{y}^{(2)}} \underbrace{u(\tilde{y}^{(2)}) y^{(3)}}_{=\tilde{y}^{(3)}} \dots \cdot y^{(r)} z^{(1)} \dots z^{(s)} \\
&= \dots \\
&= \pm \hat{y}^{(1)} \dots \hat{y}^{(r)} \underbrace{u(\tilde{y}^{(r)}) z^{(1)}}_{=\tilde{z}^{(1)}} \dots z^{(s)} \\
&= \pm \hat{y}^{(1)} \dots \hat{y}^{(r)} \underbrace{\tilde{z}^{(1)} u(\tilde{z}^{(1)})}_{=\hat{z}^{(1)}} \underbrace{u(\tilde{z}^{(1)}) z^{(2)}}_{=\tilde{z}^{(2)}} \dots z^{(s)} \\
&= \dots \\
&= \pm \hat{y}^{(1)} \dots \hat{y}^{(r)} \hat{z}^{(1)} \dots \hat{z}^{(s-1)} \underbrace{u(\tilde{z}^{(s-1)}) z^{(s)}}_{=\tilde{z}^{(s)}} \\
&= \pm \hat{y}^{(1)} \dots \hat{y}^{(r)} \hat{z}^{(1)} \dots \hat{z}^{(s)}
\end{aligned}$$

It follows that also $\hat{y}^{(r)}$ and $\hat{z}^{(s)}$ have type o_0. After replacing those $\hat{y}^{(i)}$ and $\hat{z}^{(j)}$ satisfying $\mathrm{Re}(\hat{y}^{(i)}) < 0$ and $\mathrm{Re}(\hat{z}^{(j)}) < 0$ by $-\hat{y}^{(i)}$ and $-\hat{z}^{(j)}$, respectively, we can assume that moreover $\mathrm{Re}(\hat{y}^{(1)}) > 0, \dots, \mathrm{Re}(\hat{y}^{(r)}) > 0, \mathrm{Re}(\hat{z}^{(1)}) > 0, \dots, \mathrm{Re}(\hat{z}^{(s)}) > 0$ and still $x = \pm \hat{y}^{(1)} \dots \hat{y}^{(r)} \hat{z}^{(1)} \dots \hat{z}^{(s)}$. But now,

$$\psi(x) = \psi(\pm \hat{y}^{(1)} \dots \hat{y}^{(r)} \hat{z}^{(1)} \dots \hat{z}^{(s)}) = \psi(\hat{y}^{(1)}) \dots \psi(\hat{y}^{(r)}) \psi(\hat{z}^{(1)}) \dots \psi(\hat{z}^{(s)}),$$

where $\psi(\hat{y}^{(1)}), \dots, \psi(\hat{y}^{(r)}) \in E_h$ and $\psi(\hat{z}^{(1)}), \dots, \psi(\hat{z}^{(s)}) \in E_v$.

(4) See [44, Corollary 3.2] or [43, Corollary 2.1.10].

(5) $\langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle \cong F_{\frac{p+1}{2}}$ follows directly from the uniqueness statement in (4), using

$$E_h = \psi(\{\alpha_1, \ldots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \ldots, \overline{\alpha_1}\}).$$

$\langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle \cong F_{\frac{l+1}{2}}$ follows analogously.

□

The following proposition is motivated by [43]. Some parts of our proposition are obvious generalizations of results appearing in [43], nevertheless, we try to give very detailed proofs here.

Proposition 48. *Let* $p, l \equiv 1 \pmod 4$ *be two distinct primes and* $G_{p,l} := U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))$. *Then*

(1) $\Gamma_{p,l} \subsetneqq G_{p,l}/ZG_{p,l}$. *More precisely,* $\Gamma_{p,l}$ *is (isomorphic to) a normal subgroup of* $G_{p,l}/ZG_{p,l}$ *of index* 4 *such that* $(G_{p,l}/ZG_{p,l})/\Gamma_{p,l} \cong \mathbb{Z}_2^2$.

(2) $\Gamma_{p,l} < \mathrm{SO}_3(\mathbb{Q}) < \mathrm{SO}_3(\mathbb{R}) < \mathrm{PGL}_2(\mathbb{C})$, *in particular* $\Gamma_{p,l}$ *is residually finite.*

(3) *If* q *is an odd prime different from* p *and* l, *then there is a non-trivial homomorphism* $\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$.

(4) *Let* q *be an odd prime different from* p *and* l, *and* $\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ *as constructed in (3). Then*

$$\tau(\Gamma_{p,l}) = \begin{cases} \mathrm{PSL}_2(q), & \text{if } \left(\frac{p}{q}\right) = \left(\frac{l}{q}\right) = 1 \\ \mathrm{PGL}_2(q), & \text{else.} \end{cases}$$

(5) *Let* q *be an odd prime different from* p *and* l, *and* $\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ *as in (3) and (4). Then* $\tau(a_1^2) \in \tau(\langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle)$.

Proof. (1) To simplify notation, let $G_p := U(\mathbb{H}(\mathbb{Q}_p))$. Since $ZG_{p,l} = G_{p,l} \cap ZG_p = G_{p,l} \cap ZG_l$, and $\mathbb{Z}[1/p, 1/l]$ is a subring of $\mathbb{Q}_p$ and $\mathbb{Q}_l$ (in particular $G_{p,l} \subset G_p$ and $G_{p,l} \subset G_l$), there is an injective diagonal homomorphism

$$\begin{aligned} G_{p,l}/ZG_{p,l} &\to G_p/ZG_p \times G_l/ZG_l \\ xZG_{p,l} &\mapsto (xZG_p, xZG_l). \end{aligned}$$

The isomorphism $\mathbb{H}(\mathbb{Q}_p) \to M_2(\mathbb{Q}_p)$ of Lemma 45 (with $i_p^2 + 1 = 0$) induces an isomorphism $U(\mathbb{H}(\mathbb{Q}_p)) = G_p \to \mathrm{GL}_2(\mathbb{Q}_p) = U(M_2(\mathbb{Q}_p))$ and consequently an isomorphism

$$\begin{aligned} G_p/ZG_p &\to \mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)/Z\mathrm{GL}_2(\mathbb{Q}_p) \\ xZG_p &\mapsto \left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right]. \end{aligned}$$

Let ρ be the injective composition homomorphism

$$G_{p,l}/ZG_{p,l} \hookrightarrow G_p/ZG_p \times G_l/ZG_l \xrightarrow{\cong} \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

explicitly given by

$$xZG_{p,l} \mapsto \left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right) = \tilde{\psi}(x),$$

where $x = x_0 + x_1 i + x_2 j + x_3 k \in G_{p,l}$ and $\tilde{\psi}$ is the natural extension of ψ from $\mathbb{H}(\mathbb{Z})$ to $\mathbb{H}(\mathbb{Z}[1/p, 1/l])$.

Note that

$$U(\mathbb{Z}[1/p, 1/l]) = \{\pm p^r l^s : r, s \in \mathbb{Z}\},$$

hence by Lemma 39

$$G_{p,l} = \{x \in \mathbb{H}(\mathbb{Z}[1/p, 1/l]) : |x|^2 = p^r l^s;\, r, s \in \mathbb{Z}\}$$

and by Lemma 40(2)

$$ZG_{p,l} = \{x \in \mathbb{H}(\mathbb{Z}[1/p, 1/l]) : x = \mathrm{Re}(x) = \pm p^r l^s;\, r, s \in \mathbb{Z}\}.$$

Now let $x \in \mathbb{H}(\mathbb{Z})$ be an integer quaternion such that $|x|^2 = p^r l^s$ for some $r, s \in \mathbb{N}_0$, then $x \in G_{p,l}$ and $\psi(x) = \tilde{\psi}(x) = \rho(xZG_{p,l}) \in \rho(G_{p,l}/ZG_{p,l})$, hence $\Gamma_{p,l} < \rho(G_{p,l}/ZG_{p,l}) \cong G_{p,l}/ZG_{p,l}$.

Note that each element in $G_{p,l}/ZG_{p,l}$ has a representative $xZG_{p,l}$ such that $x \in \mathbb{H}(\mathbb{Z})$ and $|x|^2 = p^r l^s; r, s \in \mathbb{N}_0$ by multiplying with large enough positive powers of p and l, however $\Gamma_{p,l} \neq \rho(G_{p,l}/ZG_{p,l})$ since x must have type o_0 in the definition of $\Gamma_{p,l}$. More precisely, we can write

$$\rho(G_{p,l}/ZG_{p,l}) = g_0\Gamma_{p,l} \sqcup g_1\Gamma_{p,l} \sqcup g_2\Gamma_{p,l} \sqcup g_3\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

where for each $\iota \in \{0, 1, 2, 3\}$ we choose $g_\iota = \psi(x)$ for some $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$ of type o_ι and norm $|x|^2 = p^r l^s; r, s \in \mathbb{N}_0$. For example, the simplest choice is to take $r = s = 0$ (i.e. $|x|^2 = 1$) and consequently

$$\begin{aligned}
g_0 &:= \psi(1) = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\right), \\
g_1 &:= \psi(i) = \left(\left[\begin{pmatrix} i_p & 0 \\ 0 & -i_p \end{pmatrix}\right], \left[\begin{pmatrix} i_l & 0 \\ 0 & -i_l \end{pmatrix}\right]\right), \\
g_2 &:= \psi(j) = \left(\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\right), \\
g_3 &:= \psi(k) = \left(\left[\begin{pmatrix} 0 & i_p \\ i_p & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & i_l \\ i_l & 0 \end{pmatrix}\right]\right).
\end{aligned}$$

To see the decomposition of $\rho(G_{p,l}/ZG_{p,l})$ given above, we observe that $p^r l^s \equiv 1 \pmod 4$, since $p, l \equiv 1 \pmod 4$ and that therefore each decomposition of $|x|^2 = p^r l^s$ as a sum of four squares is a sum of squares of three even numbers and one odd number (cf. Lemma 44(3)). If we take the quaternion multiplication on the four classes of quaternions of type o_0, o_1, o_2 and o_3 respectively, then we get a group structure, where the class of type o_0 quaternions is the identity element. The group is isomorphic to $\mathbb{Z}_2^2$, as it is seen in the following multiplication table:

$\cdot$	type o_0	type o_1	type o_2	type o_3
type o_0	type o_0	type o_1	type o_2	type o_3
type o_1	type o_1	type o_0	type o_3	type o_2
type o_2	type o_2	type o_3	type o_0	type o_1
type o_3	type o_3	type o_2	type o_1	type o_0

Because of $\psi(xy) = \psi(x)\psi(y)$, this group structure carries over to the cosets

$$\{g_0\Gamma_{p,l}, g_1\Gamma_{p,l}, g_2\Gamma_{p,l}, g_3\Gamma_{p,l}\}$$

in $\rho(G_{p,l}/ZG_{p,l})$ and we are done.

To summarize, we have shown that

$$\Gamma_{p,l} \overset{4}{\triangleleft} \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}),\, |x|^2 = p^r l^s;\, r,s \in \mathbb{N}_0\} = \rho(G_{p,l}/ZG_{p,l}) \cong G_{p,l}/ZG_{p,l}.$$

(2) If G is a group, we denote here by G/Z the quotient G/ZG of G by its center ZG. We study the following diagram of group homomorphisms:

$$\begin{tikzcd}
\Gamma_{p,l} \arrow[r] & U(\mathbb{H}(\mathbb{Z}[1/p,1/l]))/Z \arrow[r] \arrow[d, equal] & U(\mathbb{H}(\mathbb{Q}))/Z \arrow[r] \arrow[d, "\cong"] & U(\mathbb{H}(\mathbb{R}))/Z \arrow[r] \arrow[d, "\cong"] & U(\mathbb{H}(\mathbb{C}))/Z \arrow[d, "\cong"] \\
 & G_{p,l}/Z & \mathrm{SO}_3(\mathbb{Q}) & \mathrm{SO}_3(\mathbb{R}) & \mathrm{PGL}_2(\mathbb{C})
\end{tikzcd}$$

The homomorphisms in the top line are all injective: the first one $\Gamma_{p,l} \to G_{p,l}/Z$ is described in part (1) of this proposition. The other three homomorphisms are induced by the natural injective group homomorphisms (which are induced themselves by the chain of subrings $\mathbb{Z}[1/p,1/l]) \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$)

$$U(\mathbb{H}(\mathbb{Z}[1/p,1/l])) \hookrightarrow U(\mathbb{H}(\mathbb{Q})) \hookrightarrow U(\mathbb{H}(\mathbb{R})) \hookrightarrow U(\mathbb{H}(\mathbb{C})), \tag{18}$$

since

$$ZU(\mathbb{H}(\mathbb{Z}[1/p,1/l])) \subset ZU(\mathbb{H}(\mathbb{Q})) \subset ZU(\mathbb{H}(\mathbb{R})) \subset ZU(\mathbb{H}(\mathbb{C})). \tag{19}$$

Assertion (19) follows directly from (18), using the fact (see Lemma 40(2))

$$ZU(\mathbb{H}(R)) = U(\mathbb{H}(R)) \cap \{x \in U(\mathbb{H}(\mathbb{C})) : x = \mathrm{Re}(x)\},$$

which holds if $R \in \{\mathbb{Z}[1/p,1/l], \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

The homomorphisms

$$U(\mathbb{H}(\mathbb{Z}[1/p,1/l]))/Z \longrightarrow U(\mathbb{H}(\mathbb{Q}))/Z \longrightarrow U(\mathbb{H}(\mathbb{R}))/Z \longrightarrow U(\mathbb{H}(\mathbb{C}))/Z$$

are injective, since (18) directly implies $U(\mathbb{H}(R_1)) \cap ZU(\mathbb{H}(R_2)) < ZU(\mathbb{H}(R_1))$, whenever $(R_1,R_2) \in \{(\mathbb{Z}[1/p,1/l],\mathbb{Q}), (\mathbb{Q},\mathbb{R}), (\mathbb{R},\mathbb{C})\}$. In fact, the equality $U(\mathbb{H}(R_1)) \cap ZU(\mathbb{H}(R_2)) = ZU(\mathbb{H}(R_1))$ holds by (19).

To get $U(\mathbb{H}(\mathbb{Q}))/Z \cong \mathrm{SO}_3(\mathbb{Q})$, first note that $U(\mathbb{H}(\mathbb{Q})) = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$ and define

$$\begin{aligned}\vartheta : U(\mathbb{H}(\mathbb{Q})) &\to \mathrm{SO}_3(\mathbb{Q})\\ x &\mapsto \frac{1}{|x|^2}\begin{pmatrix} x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{pmatrix},\end{aligned}$$

where $x = x_0 + x_1 i + x_2 j + x_3 k \in U(\mathbb{H}(\mathbb{Q}))$. It is well-known that ϑ is a surjective group homomorphism. Even the restricted map

$$\vartheta|_{\mathbb{H}(\mathbb{Z})\setminus\{0\}} : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{SO}_3(\mathbb{Q})$$

is surjective, since $\vartheta(ax) = \vartheta(x)$, if $a \in \mathbb{Q}^\times$, $x \in U(\mathbb{H}(\mathbb{Q}))$. For an elementary proof of the surjectivity of $\vartheta|_{\mathbb{H}(\mathbb{Z})\setminus\{0\}}$, see [41]. Moreover, it is easy to check that

$$\ker(\vartheta) = \{x \in \mathbb{H}(\mathbb{Q}) \setminus \{0\} : x = \mathrm{Re}(x)\} = ZU(\mathbb{H}(\mathbb{Q})).$$

Note that the axis of the rotation $\vartheta(x) \in \mathrm{SO}_3(\mathbb{Q})$ is $(x_1, x_2, x_3)^T$ and the rotation angle ω satisfies

$$\cos\omega = \frac{x_0^2 - x_1^2 - x_2^2 - x_3^2}{|x|^2}.$$

To prove $U(\mathbb{H}(\mathbb{R}))/Z \cong \mathrm{SO}_3(\mathbb{R})$, replace $\mathbb{Q}$ by $\mathbb{R}$ above.

The isomorphism $U(\mathbb{H}(\mathbb{C}))/Z \cong \mathrm{PGL}_2(\mathbb{C})$ follows directly from Lemma 45.

Note that the injective composition homomorphism $\Gamma_{p,l} \to \mathrm{SO}_3(\mathbb{Q})$ is explicitly constructed as follows: if $\gamma \in \Gamma_{p,l}$ is given as $\gamma = \psi(x)$, where $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$ has type o_0 and $|x|^2 = p^r l^s; r, s \in \mathbb{N}_0$, then the image of γ in $\mathrm{SO}_3(\mathbb{Q})$ is $\vartheta(x)$, independent of the choice of x. In the same way, the image of $\gamma = \psi(x)$ in $\mathrm{PGL}_2(\mathbb{C})$ is

$$\left[\begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix}\right].$$

By a result of Malcev ([50]), finitely generated linear groups (over a field of characteristic zero) are residually finite.

(3) Let q be an odd prime different from p, l and let $G_{q,p,l} := U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l]))$. As in the proof of (2), we denote by G/Z the quotient G/ZG of G by its center ZG. We want to define the homomorphism $\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ as composition

$$\Gamma_{p,l} \hookrightarrow G_{p,l}/Z \to G_{q,p,l}/Z \xrightarrow{\cong} U(\mathbb{H}(\mathbb{Z}_q))/Z \xrightarrow{\cong} \mathrm{PGL}_2(q).$$

We describe now these four homomorphisms.

The injection $\Gamma_{p,l} \hookrightarrow G_{p,l}/Z$ is given by part (1) of this proposition.

The unital ring homomorphism $\mathbb{Z}[1/p, 1/l] \to \mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l]$ extends to a unital ring homomorphism $\mathbb{H}(\mathbb{Z}[1/p, 1/l]) \to \mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l])$ mapping 1 to 1, i to i, j to j and k to k (see [22, Chapter 2.4]) and induces a group homomorphism of the invertible elements $G_{p,l} \to G_{q,p,l}$. It is easy to see that the image of $ZG_{p,l}$ is contained in $ZG_{q,p,l}$. This gives the second homomorphism

$$G_{p,l}/Z \to G_{q,p,l}/Z.$$

The map

$$\begin{aligned} \phi : \mathbb{Z}_q &\to \mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l] \\ v + q\mathbb{Z} &\mapsto v + q\mathbb{Z}[1/p, 1/l], \end{aligned}$$

$v \in \mathbb{Z}$, is an isomorphism of rings (or fields, since q is prime), and ϕ^{-1} therefore induces isomorphisms

$$\mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l]) \xrightarrow{\cong} \mathbb{H}(\mathbb{Z}_q),$$
$$G_{q,p,l} = U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l])) \xrightarrow{\cong} U(\mathbb{H}(\mathbb{Z}_q))$$

and finally an isomorphism $G_{q,p,l}/Z \to U(\mathbb{H}(\mathbb{Z}_q))/Z$. The only non-trivial thing to check is the surjectivity of ϕ: First, we have $\phi(0 + q\mathbb{Z}) = 0 + q\mathbb{Z}[1/p, 1/l]$. Now, take any element

$$tp^r l^s + q\mathbb{Z}[1/p, 1/l] \in \mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l],$$

where $t \in \mathbb{Z} \setminus \{0\}$ and t is relatively prime to p and l. We assume that $r, s < 0$ (if $r, s \geq 0$, then $\phi^{-1}(tp^r l^s + q\mathbb{Z}[1/p, 1/l]) = tp^r l^s + q\mathbb{Z}$; in the cases $r \geq 0, s < 0$ and $r < 0, s \geq 0$ the proofs are similar to the proof for the case $r, s < 0$ given now). Then $\gcd(p^{-r}l^{-s}, q) = 1$ obviously divides t, hence (see e.g. [35, Proposition 3.3.1]) there is an integer u such that $p^{-r}l^{-s}u \equiv t \pmod q$, i.e. $t - p^{-r}l^{-s}u \in q\mathbb{Z}$ and

$$tp^r l^s - u = p^r l^s(t - p^{-r}l^{-s}u) \in q\mathbb{Z}[1/p, 1/l].$$

This implies

$$tp^r l^s + q\mathbb{Z}[1/p, 1/l] = u + q\mathbb{Z}[1/p, 1/l] = \phi(u + q\mathbb{Z}).$$

The isomorphism $U(\mathbb{H}(\mathbb{Z}_q))/Z \cong \mathrm{PGL}_2(q)$ follows directly from Lemma 45, since there exist elements c and d in the field $\mathbb{Z}_q$ such that $c^2 + d^2 + 1 = 0$ in $\mathbb{Z}_q$, see [22, Proposition 2.4.3].

Therefore, if $\gamma \in \Gamma_{p,l}$ is given by $\psi(x_0+x_1 i+x_2 j+x_3 k)$ (where we require as in the definition of $\Gamma_{p,l}$ that $x \in \mathbb{H}(\mathbb{Z})$ has type o_0 and $|x|^2 = p^r l^s; r, s \in \mathbb{N}_0$) and we have chosen $c, d \in \mathbb{Z}$ such that $c^2 + d^2 + 1 \equiv 0 \pmod q$, then $\tau = \tau_{c,d} : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ is constructed as

$$\tau_{c,d}(\gamma) = \left[\begin{pmatrix} x_0 + x_1 c + x_3 d + q\mathbb{Z} & -x_1 d + x_2 + x_3 c + q\mathbb{Z} \\ -x_1 d - x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c - x_3 d + q\mathbb{Z} \end{pmatrix}\right].$$

For example if $q \equiv 1 \pmod 4$, we can choose $d = 0$ and $c \in \{1, \ldots, q-1\}$, such that $c^2+1 \equiv 0 \pmod q$, and $\tau = \tau_{c,0}$ simplifies to

$$\gamma \mapsto \left[\begin{pmatrix} x_0 + x_1 c + q\mathbb{Z} & x_2 + x_3 c + q\mathbb{Z} \\ -x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c + q\mathbb{Z} \end{pmatrix}\right].$$

What happens if we take $q = 2$?

$$G_{2,p,l} \cong U(\mathbb{H}(\mathbb{Z}_2)) \cong \mathbb{Z}_2^3$$

is abelian, hence

$$G_{2,p,l}/Z \cong U(\mathbb{H}(\mathbb{Z}_2))/Z = 1 \neq \mathrm{PGL}_2(2) \cong S_3.$$

Note that the field $\mathbb{Z}_2$ is excluded in the assumptions of Lemma 45.

(4) First, we show that $\tau(\Gamma_{p,l}) < \mathrm{PSL}_2(q)$ if and only if $\left(\frac{p}{q}\right) = \left(\frac{l}{q}\right) = 1$. The group $\Gamma_{p,l}$ is generated by $\{a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}\}$, hence $\tau(\Gamma_{p,l}) < \mathrm{PSL}_2(q)$ if and only if

$$\{\tau(a_1), \ldots, \tau(a_{\frac{p+1}{2}}), \tau(b_1), \ldots, \tau(b_{\frac{l+1}{2}})\} \subset \mathrm{PSL}_2(q).$$

Since $\tau(a_1), \ldots, \tau(a_{\frac{p+1}{2}})$ are represented by matrices in $\mathrm{GL}_2(q)$ with determinant $p+q\mathbb{Z} \in \mathbb{Z}_q$ and $\tau(b_1), \ldots, \tau(b_{\frac{l+1}{2}})$ are represented by matrices in $\mathrm{GL}_2(q)$ with determinant $l + q\mathbb{Z} \in \mathbb{Z}_q$, the condition $\tau(\Gamma_{p,l}) < \mathrm{PSL}_2(q)$ is by Lemma 42 equivalent to the condition $\{p+q\mathbb{Z}, l+q\mathbb{Z}\} \subset (\mathbb{Z}_q^\times)^2$. But this is equivalent to $\left(\frac{p}{q}\right) = \left(\frac{l}{q}\right) = 1$ by Lemma 43.

By [43, Lemma 7.4.2] or [44, Proposition 3.3], we have

$$\mathrm{PSL}_2(q) < \tau(\langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle) \text{ and } \mathrm{PSL}_2(q) < \tau(\langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle),$$

in particular $\mathrm{PSL}_2(q) < \tau(\Gamma_{p,l})$.

The statement follows now directly since $[\mathrm{PGL}_2(q) : \mathrm{PSL}_2(q)] = 2$.

(5) Exactly as in (4), we can show that

$$\tau(\langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle) = \begin{cases} \mathrm{PSL}_2(q), & \text{if } \left(\frac{l}{q}\right) = 1 \\ \mathrm{PGL}_2(q), & \text{if } \left(\frac{l}{q}\right) = -1 \end{cases}$$

Since $\tau(a_1^2) = \tau(a_1)^2$ is represented by a matrix in $\mathrm{GL}_2(q)$ with determinant $(p + q\mathbb{Z})^2 = p^2 + q\mathbb{Z} \in \mathbb{Z}_q$, we have $\tau(a_1^2) \in \mathrm{PSL}_2(q)$ by Lemma 42 and the claim follows.

□

See Table 16 for some information about the group $U(\mathbb{H}(R))/ZU(\mathbb{H}(R))$, where R is a commutative ring with unit, $p, l \equiv 1 \pmod 4$ are distinct primes and q is an odd prime.

R	$U(\mathbb{H}(R))/ZU(\mathbb{H}(R))$
$\mathbb{Z}[1/p, 1/l]$	contains $\Gamma_{p,l}$ as index 4 subgroup
$\mathbb{Z}[1/p]$	important in [43], virtually $F_{\frac{p+1}{2}}$
$\mathbb{Z}$	$\mathbb{Z}_2^2$
$\mathbb{Z}_q$	$\mathrm{PGL}_2(q)$
$\mathbb{Z}_2$	1
$\mathbb{Q}$	$\mathrm{SO}_3(\mathbb{Q})$
$\mathbb{R}$	$\mathrm{SO}_3(\mathbb{R})$
$\mathbb{C}$	$\mathrm{PGL}_2(\mathbb{C})$
$\mathbb{Q}_q$	$\mathrm{PGL}_2(\mathbb{Q}_q)$

Table 16: $U(\mathbb{H}(R))/ZU(\mathbb{H}(R))$ for some rings R.

The following result is also mentioned in [60, Example 5.12] and [29, Proposition 3.2 and Proof of Theorem 4.1]. It is a very special case of Proposition 74(3).

Proposition 49. *$\Gamma_{p,l}$ contains a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. By Lemma 44(1), we can choose $x = x_0 + x_1 i$, $y = y_0 + y_1 i \in \mathbb{H}(\mathbb{Z})$ such that x_0, y_0 are odd, x_1, y_1 are even, $|x|^2 = x_0^2 + x_1^2 = p$, $|y|^2 = y_0^2 + y_1^2 = l$. Obviously, we have $xy = yx$, hence $\psi(x)\psi(y) = \psi(y)\psi(x)$. The subgroup $\langle \psi(x), \psi(y) \rangle$ is isomorphic to $\mathbb{Z}^2$, using the same arguments as in the proof of Corollary 58. □

Jason S. Kimberley and Guyan Robertson have computed presentations of $\Gamma_{p,l}$ for many pairs (p, l). They conjecture for the abelianization $\Gamma_{p,l}^{ab}$

Conjecture 24. *(Kimberley-Robertson [39, Section 6]) Let $p, l \equiv 1 \pmod 4$ be two distinct primes, then*

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4^3, & \text{if } r = 1, \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2, & \text{if } r = 2, \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3, & \text{if } r = 3, \\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \text{if } r = 6, \end{cases}$$

where

$$r = \gcd\left(\frac{p-1}{4}, \frac{l-1}{4}, 6\right).$$

Note that the smallest pairs (p, l) such that $r = 1, 2, 3, 6$ are $(5, 13)$, $(17, 41)$, $(13, 37)$, $(73, 97)$ respectively. Conjecture 24 is equivalent to the following conjecture (see Section 5.5 for generalizations of Conjecture 25):

Conjecture 25. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes.*
If $p, l \equiv 1 \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \text{if } p, l \equiv 1 \pmod 3, \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2, & \text{else.} \end{cases}$$

If $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$, then

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3, & \text{if } p, l \equiv 1 \pmod 3, \\ \mathbb{Z}_2 \times \mathbb{Z}_4^3, & \text{else.} \end{cases}$$

Proof of the equivalence of Conjecture 24 and Conjecture 25. Let $p, l \equiv 1 \pmod 4$ be two distinct primes. First note that $r \in \{1, 2, 3, 6\}$ in Conjecture 24 and that all possibilities for (p, l) are treated in Conjecture 25.

If $r = 6$, then $(p-1)/4 = 6s$ and $(l-1)/4 = 6t$ for some $s, t \in \mathbb{N}$, i.e. $p = 24s + 1$ and $l = 24t + 1$. It follows $p, l \equiv 1 \pmod 8$, $p, l \equiv 1 \pmod 3$.

If $r = 3$, then $(p-1)/4 = 3s$ and $(l-1)/4 = 3t$, where s or t is odd (otherwise r would be 6). Consequently, we have $p = 12s + 1$ and $l = 12t + 1$, in particular $p, l \equiv 1 \pmod 3$. If s is odd, then $p \equiv 5 \pmod 8$. If t is odd, then $l \equiv 5 \pmod 8$.

If $r = 2$, then $(p-1)/4 = 2s$ and $(l-1)/4 = 2t$, i.e. $p = 8s + 1$ and $l = 8t + 1$, hence $p, l \equiv 1 \pmod 8$. Moreover, $s \not\equiv 0 \pmod 3$ or $t \not\equiv 0 \pmod 3$ (otherwise r would be 6). In the first case, we have $p \not\equiv 1 \pmod 3$, in the second case $l \not\equiv 1 \pmod 3$.

If $r = 1$, then $(p-1)/4 = 2s + 1$ or $(l-1)/4 = 2t + 1$ (otherwise r would be even), hence $p = 8s + 5$ or $l = 8t + 5$, i.e. $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$. Moreover: $(p-1)/4 = 3s + 1$ or $(p-1)/4 = 3s + 2$ or $(l-1)/4 = 3t + 1$ or $(l-1)/4 = 3s + 2$ for some $s, t \in \mathbb{N}_0$ (otherwise r would be a multiple of 3), hence $p = 12s + 5$ or $p = 12s + 9$ or $l = 12t + 5$ or $l = 12t + 9$, in particular $p \not\equiv 1 \pmod 3$ or $l \not\equiv 1 \pmod 3$. $\square$

The structure of $\Gamma_{p,l}^{ab}$ also seems to depend only on the number of commuting quaternions whose ψ-images generate $\Gamma_{p,l}$. To make this precise, if $l \equiv 1 \pmod 4$ is prime, let $Y_l \subset \mathbb{H}(\mathbb{Z})$ be any set of cardinality $\frac{l+1}{2}$, such that $\langle \psi(Y_l) \rangle \cong F_{\frac{l+1}{2}}$ and each element $y = y_0 + y_1 i + y_2 j + y_3 k \in Y_l$ has type o_0 and satisfies $y_0 > 0$, $|y|^2 = l$. We think of $Y_l = \{\psi^{-1}(b_1), \dots, \psi^{-1}(b_{\frac{l+1}{2}})\}$ and $Y_p = \{\psi^{-1}(a_1), \dots, \psi^{-1}(a_{\frac{p+1}{2}})\}$, where

$$\Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2) \rangle.$$

Let

$$c_{p,l} := |\{(x, y) : x \in Y_p;\ y \in Y_l;\ xy = yx\}|.$$

Note that the definition of $c_{p,l}$ is independent of the explicit choice of the elements in Y_p and Y_l. Obviously,

$$c_{p,l} \le \min \left\{ \frac{p+1}{2}, \frac{l+1}{2} \right\}.$$

Moreover, $c_{p,l} \ge 3$, since Y_p contains elements of the form $x_0 + x_1 i$, $x_0 + x_2 j$, $x_0 + x_3 k$ and Y_l contains elements of the form $y_0 + y_1 i$, $y_0 + y_2 j$, $y_0 + y_3 k$.

Conjecture 26. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes, and*

$$r = \gcd\left(\frac{p-1}{4}, \frac{l-1}{4}, 6\right)$$

as in Conjecture 24. Then

$$c_{p,l} \equiv \begin{cases} 3 \pmod{12}, & \text{if } r = 1, \\ 9 \pmod{12}, & \text{if } r = 2, \\ 7 \pmod{12}, & \text{if } r = 3, \\ 1 \pmod{12}, & \text{if } r = 6. \end{cases}$$

We have checked Conjecture 26 for all possible $p, l < 1000$. The following values for $c_{p,l}$ appear in this range:

$$c_{p,l} \in \begin{cases} \{3, 15, 27, 39, 51, 63, 75, 87, 99\}, & \text{if } r = 1, \\ \{9, 21, 33, 45, 57, 69, 81, 93, 105, 117, 129, 153\}, & \text{if } r = 2, \\ \{7, 19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 151\}, & \text{if } r = 3, \\ \{37, 49, 61, 73, 85, 97, 109, 121, 133\}, & \text{if } r = 6. \end{cases}$$

See Table 17 for the frequencies of the values of $c_{p,l}$, where $p, l \equiv 1 \pmod 4$ are primes such that $p < l < 1000$.

$c_{p,l}$	3	15	27	39	51	63	75	87	99	111	123	135	147	
	1242	449	143	56	34	17	7	5	2					1955
$c_{p,l}$	9	21	33	45	57	69	81	93	105	117	129	141	153	
	178	158	84	57	40	21	8	9	12	5	2		1	575
$c_{p,l}$	7	19	31	43	55	67	79	91	103	115	127	139	151	
	236	130	79	42	18	8	12	6	1	4	2		1	539
$c_{p,l}$	1	13	25	37	49	61	73	85	97	109	121	133	145	
				26	15	15	16	7	4	3	2	3		91

Table 17: $c_{p,l}$ and its frequency, $p < l < 1000$.

Combining Conjecture 26 with Conjecture 24, we get

Conjecture 27. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes, then*

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4^3, & \text{if } c_{p,l} \equiv 3 \pmod{12}, \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2, & \text{if } c_{p,l} \equiv 9 \pmod{12}, \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3, & \text{if } c_{p,l} \equiv 7 \pmod{12}, \\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \text{if } c_{p,l} \equiv 1 \pmod{12}. \end{cases}$$

Now, we want to prove that $\Gamma_{p,l}$ is commutative transitive. This has some nice applications to centralizers of powers, or to detect anti-tori in $\Gamma_{p,l}$ (see Proposition 57 in Section 5.6).

Lemma 50. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes. Let $x, y \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|x|^2, |y|^2 \in \{p^r l^s : r, s \in \mathbb{N}_0\}$. Then $xy = yx$ if and only if $\psi(x)\psi(y) = \psi(y)\psi(x)$.*

Proof. Obviously $xy = yx$ implies $\psi(x)\psi(y) = \psi(y)\psi(x)$. Assume now $\psi(x)\psi(y) = \psi(y)\psi(x)$. Then $\psi(xy) = \psi(yx)$ and $xy = \lambda yx$ for some $\lambda \in \mathbb{Q}^\times$. Taking the norm $|\cdot|^2$ of $xy = \lambda yx$, we conclude $\lambda^2 = 1$, hence $\lambda = 1$ or $\lambda = -1$. If $\lambda = 1$, then $xy = yx$ and we are done. The case $\lambda = -1$ is impossible since $xy = -yx$ together with $\mathrm{Re}(x) \neq 0$ implies by Lemma 41(3) the contradiction $y = 0$. □

Proposition 51. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes. Then $\Gamma_{p,l}$ is commutative transitive, i.e. if $x, y, z \in \mathbb{H}(\mathbb{Z})$ are of type o_0 such that $x \neq \mathrm{Re}(x)$, $y \neq \mathrm{Re}(y)$, $z \neq \mathrm{Re}(z)$ and $|x|^2, |y|^2, |z|^2 \in \{p^r l^s : r, s \in \mathbb{N}_0\}$, $\psi(x)\psi(y) = \psi(y)\psi(x)$ and $\psi(x)\psi(z) = \psi(z)\psi(x)$, then also $\psi(y)\psi(z) = \psi(z)\psi(y)$. In other words, the relation of commutativity is transitive on the non-trivial elements of $\Gamma_{p,l}$.*

Proof. Note that for x of type o_0 we have $x \neq \mathrm{Re}(x)$, if and only if $\psi(x) \neq 1$. By Lemma 50, we have $xy = yx$ and $xz = zx$. Moreover, again by Lemma 50, $\psi(y)\psi(z) = \psi(z)\psi(y)$ if and only if $yz = zy$. But $yz = zy$ follows now directly by Lemma 41(4). □

Corollary 52. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes, $\Gamma = \Gamma_{p,l}$ and $w \in \Gamma \setminus \{1\}$ a non-trivial element.*

(1) *If $t \in \mathbb{N}$, then $Z_\Gamma(w^t) = Z_\Gamma(w)$.*

(2) *$Z_\Gamma(w)$ is abelian.*

(3) $Z\Gamma = 1$.

Proof. (1) Since w and w^t commute, the statement follows directly from Proposition 51 (using the fact that Γ is torsion-free).

(2) Again, this is a direct consequence of Proposition 51.

(3) Of course, it follows from the general result Corollary 7(3) for $(2m, 2n)$-groups. Here, it follows from Proposition 51, since $1 \neq x \in Z\Gamma$ implies that Γ is abelian.

$\square$

Using the following proposition due to Shahar Mozes ([53]) together with Proposition 8, we give some applications to number theory, illustrated for two concrete examples:

Proposition 53. *(Mozes [53, Proposition 3.15]) Let $p, l \equiv 1 \pmod 4$ be two distinct primes,*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2) \rangle$$

and let $z = z_0 + z_1 i + z_2 j + z_3 k \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $z \neq z_0$ and $|z|^2 = l^s$ for some $s \in \mathbb{N}$. Take $c_1, c_2, c_3 \in \mathbb{Z}$ relatively prime such that $c := c_1 i + c_2 j + c_3 k$ commutes with z. Then there exists an element $a \in \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle \setminus \{1\} \subset \Gamma$ commuting with $\psi(z)$ if and only if there are integers $x, y \in \mathbb{Z}$ such that $\gcd(x, y) = \gcd(x, pl) = \gcd(y, pl) = 1$ and $x^2 + 4|c|^2 y^2 \in \{p^r l^s : r, s \in \mathbb{N}\}$.

Proposition 54. (1) *There are no pairs $x, y \in \mathbb{Z}$ such that*

$$\gcd(x, y) = \gcd(x, 65) = \gcd(y, 65) = 1$$

and

$$x^2 + 12y^2 \in \{5^r 13^s : r, s \in \mathbb{N}\}.$$

(2) *There are no pairs $x, y \in \mathbb{Z}$ such that*

$$\gcd(x, y) = \gcd(x, 221) = \gcd(y, 221) = 1$$

and

$$x^2 + 8y^2 \in \{13^r 17^s : r, s \in \mathbb{N}\}.$$

Proof. (1) For $b_1 = \psi(1 + 2i + 2j + 2k) \in \Gamma_{5,13} =: \Gamma$ we have $Z_\Gamma(b_1) = \langle b_1 \rangle$, see Theorem 41(8) below. In particular, b_1 does not commute with any element in $\langle a_1, a_2, a_3 \rangle \setminus \{1\}$. The statement follows now by Proposition 53, taking $c = i + j + k$.

(2) By Theorem 40(4) below: $Z_\Gamma(b_4) = \langle b_4 \rangle$, where $b_4 = \psi(3 + 2i + 2j) \in \Gamma_{13,17} =: \Gamma$. Taking $c = i + j$, we can again apply Proposition 53.

$\square$

The results on centralizers in $\Gamma_{p,l}$ used in the proof of the preceding proposition can also be applied to give statements about non-commuting quaternions. We first illustrate it for $(p, l) \in \{(5, 13), (13, 17)\}$.

Proposition 55. (1) *Let $y = 1 + 2i + 2j + 2k$. Then there is no $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$, $x \neq \mathrm{Re}(x)$ of type o_0 such that $|x|^2 \in \{5^r : r \in \mathbb{N}\}$ and $xy = yx$.*

(2) *Let $y = 3 + 2i + 2j$. Then there is no $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$, $x \neq \mathrm{Re}(x)$ of type o_0 such that $|x|^2 \in \{13^r : r \in \mathbb{N}\}$ and $xy = yx$.*

Proof. (1) Let $\Gamma = \Gamma_{5,13}$ and $b_1 = \psi(y) \in \Gamma$. Assume that $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$ is of type o_0 such that $|x|^2 \in \{5^r : r \in \mathbb{N}\}$ and $xy = yx$, where $x \neq \mathrm{Re}(x)$. This implies $\psi(x) \neq 1$ and $\psi(x) \in Z_\Gamma(b_1)$, contradicting $Z_\Gamma(b_1) = \langle b_1 \rangle$ (Theorem 41(8)).

(2) Same proof taking $p = 13$, $l = 17$, $b_4 = \psi(y) \in \Gamma = \Gamma_{13,17}$ and using $Z_\Gamma(b_4) = \langle b_4 \rangle$ (Theorem 40(4)).

□

Here is the general statement:

Proposition 56. *Let $p, l \equiv 1 \pmod 4$ be two distinct primes and*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2) \rangle.$$

Assume that $\rho_v(b_j)(a) \neq a$ for some $b_j \in \{b_1, \dots, b_{\frac{l+1}{2}}\}$ and all $a \in E_h$. Let $y \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|y|^2 = l$ and $b_j = \psi(y)$. Then there is no $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$, $x \neq \mathrm{Re}(x)$ of type o_0 such that $|x|^2 \in \{p^r : r \in \mathbb{N}\}$ and $xy = yx$.

Proof. As in the proof of Proposition 55 this follows directly from the fact $Z_\Gamma(b_j) = \langle b_j \rangle$ which is a consequence of Proposition 8(1b). □

Now, we want to study the two examples $\Gamma_{13,17}$ and $\Gamma_{5,13}$.

5.2.1 Example: $p = 13$, $l = 17$

Using the explicit identification

$$\begin{aligned}
a_1 &= \psi(1 + 2i + 2j + 2k), \\
a_2 &= \psi(1 + 2i + 2j - 2k), \\
a_3 &= \psi(1 + 2i - 2j + 2k), \\
a_4 &= \psi(1 - 2i + 2j + 2k), \\
a_5 &= \psi(3 + 2i), \\
a_6 &= \psi(3 + 2j), \\
a_7 &= \psi(3 + 2k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1 + 4i), \\
b_2 &= \psi(1 + 4j), \\
b_3 &= \psi(1 + 4k), \\
b_4 &= \psi(3 + 2i + 2j), \\
b_5 &= \psi(3 + 2i - 2j), \\
b_6 &= \psi(3 + 2i + 2k), \\
b_7 &= \psi(3 + 2i - 2k), \\
b_8 &= \psi(3 + 2j + 2k), \\
b_9 &= \psi(3 + 2j - 2k),
\end{aligned}$$

we get the following example $\Gamma = \Gamma_{13,17}$ (the corresponding complex X is denoted by $\mathcal{A}_{13,17}$ in [16] and essentially used there in the construction of finitely presented torsion-free simple groups, see [16, Theorem 6.4]).

Example 40. $R(7,9) = R((p+1)/2,(l+1)/2) :=$

$$\left\{\begin{array}{lllllll}
a_1b_1a_3b_3, & a_1b_2a_2b_1, & a_1b_3a_4b_2, & a_1b_4a_6b_8, & a_1b_5a_7b_1^{-1}, & a_1b_6a_5b_4, & a_1b_7a_2^{-1}b_6^{-1}, \\
a_1b_8a_7b_6, & a_1b_9a_5b_2^{-1}, & a_1b_9^{-1}a_3^{-1}b_8^{-1}, & a_1b_8^{-1}a_2^{-1}b_9, & a_1b_7^{-1}a_6b_3^{-1}, & a_1b_6^{-1}a_4^{-1}b_7^{-1}, & a_1b_5^{-1}a_4^{-1}b_4^{-1}, \\
a_1b_4^{-1}a_3^{-1}b_5, & a_1b_3^{-1}a_5b_9^{-1}, & a_1b_2^{-1}a_7b_5^{-1}, & a_1b_1^{-1}a_6b_7, & a_2b_2a_3^{-1}b_3^{-1}, & a_2b_3a_6b_6^{-1}, & a_2b_4a_5b_7, \\
a_2b_5a_4b_4^{-1}, & a_2b_6a_6b_1^{-1}, & a_2b_7a_7^{-1}b_9, & a_2b_9a_6b_4, & a_2b_9^{-1}a_4b_8^{-1}, & a_2b_8^{-1}a_5b_3, & a_2b_6^{-1}a_3b_7^{-1}, \\
a_2b_5^{-1}a_7^{-1}b_2^{-1}, & a_2b_4^{-1}a_3b_5^{-1}, & a_2b_3^{-1}a_4^{-1}b_1, & a_2b_2^{-1}a_5b_8, & a_2b_1^{-1}a_7^{-1}b_5, & a_3b_1a_4^{-1}b_2^{-1}, & a_3b_2a_5b_8^{-1}, \\
a_3b_5a_5b_6, & a_3b_6a_7b_9^{-1}, & a_3b_7a_6^{-1}b_1^{-1}, & a_3b_8a_5b_3^{-1}, & a_3b_9^{-1}a_6^{-1}b_5, & a_3b_8^{-1}a_4b_9, & a_3b_6^{-1}a_4b_7, \\
a_3b_4^{-1}a_7b_2, & a_3b_3^{-1}a_6^{-1}b_7^{-1}, & a_3b_1^{-1}a_7b_4, & a_4b_1a_7b_4^{-1}, & a_4b_4a_7b_2^{-1}, & a_4b_8a_6b_5^{-1}, & a_4b_9^{-1}a_5^{-1}b_3^{-1}, \\
a_4b_7^{-1}a_7b_8, & a_4b_6^{-1}a_6b_1, & a_4b_5^{-1}a_5^{-1}b_7^{-1}, & a_4b_3^{-1}a_6b_6, & a_4b_2^{-1}a_5^{-1}b_9, & a_5b_1a_5^{-1}b_1^{-1}, & a_5b_7^{-1}a_5b_6^{-1}, \\
a_5b_5^{-1}a_5b_4^{-1}, & a_6b_2a_6^{-1}b_2^{-1}, & a_6b_5a_6b_4^{-1}, & a_6b_9^{-1}a_6b_8^{-1}, & a_7b_3a_7^{-1}b_3^{-1}, & a_7b_7a_7b_6^{-1}, & a_7b_9a_7b_8^{-1}
\end{array}\right\}.$$

Theorem 40. (1) *$P_h \cong \mathrm{PSL}_2(13) < S_{14}$, $P_v \cong \mathrm{PSL}_2(17) < S_{18}$.*

(2) *$\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^3$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_{16}^3$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.*

(3) *Any non-trivial normal subgroup of Γ has finite index.*

(4) *$Z_\Gamma(b) = N_\Gamma(\langle b\rangle) = \langle b\rangle$, if $b \in \{b_4,\dots,b_9\}$.*
$Z_\Gamma(a) = N_\Gamma(\langle a\rangle) = \langle a\rangle$, if $a \in \{a_1,\dots,a_4\}$.

(5) *Let*
$$V := \langle 1+2i+2j+2k,\ 3+2i,\ 1+4j,\ 3+2i+2j\rangle < U(\mathbb{H}(\mathbb{Q})).$$
Then $\Gamma \cong V/ZV$.

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (1,8,13)(2,9,4)(3,6,14)(7,12,11),\\
\rho_v(b_2) &= (1,10,11)(2,7,14)(3,4,8)(5,13,12),\\
\rho_v(b_3) &= (1,9,12)(2,3,10)(4,5,14)(6,11,13),\\
\rho_v(b_4) &= (1,4,8,3,13,5,10)(2,11,7,12,14,6,9),\\
\rho_v(b_5) &= (1,8,13,4,9,6,3)(2,12,5,10,11,14,7),\\
\rho_v(b_6) &= (1,2,9,4,12,7,8)(3,13,6,11,14,5,10),\\
\rho_v(b_7) &= (1,4,5,10,2,12,9)(3,6,14,13,8,7,11),\\
\rho_v(b_8) &= (1,3,10,2,11,6,9)(4,12,5,13,14,7,8),\\
\rho_v(b_9) &= (1,10,11,3,8,7,2)(4,13,6,9,12,14,5),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,5,17,3,12,18,2,9,16)(4,14,15,6,7,13,8,10,11),\\
\rho_h(a_2) &= (1,6,3,2,14,18,16,11,17)(4,5,15,9,8,10,7,13,12),\\
\rho_h(a_3) &= (1,7,16,17,15,18,3,8,2)(4,14,10,11,9,6,12,13,5),\\
\rho_h(a_4) &= (1,3,10,17,18,13,16,2,4)(5,8,9,11,12,6,7,14,15),\\
\rho_h(a_5) &= (2,8,3,10,17,11,16,9)(4,14,6,12,5,15,7,13),\\
\rho_h(a_6) &= (1,7,16,13,18,12,3,6)(4,5,9,11,14,15,8,10),\\
\rho_h(a_7) &= (1,4,2,14,18,15,17,5)(6,7,8,9,12,13,10,11).
\end{aligned}$$

(2) We use GAP ([28]).

(3) We can apply [16, Theorem 4.1] using the results in [16, Section 2.4] and [15, Section 1.8]. Note that $\mathrm{PSL}_2(\mathbb{Q}_{13}) \subsetneqq H_1 \subsetneqq \mathrm{PGL}_2(\mathbb{Q}_{13})$ and $\mathrm{PSL}_2(\mathbb{Q}_{17}) \subsetneqq H_2 \subsetneqq \mathrm{PGL}_2(\mathbb{Q}_{17})$ such that $[\mathrm{PGL}_2(\mathbb{Q}_{13}) : H_1] = [H_1 : \mathrm{PSL}_2(\mathbb{Q}_{13})] = 2$ and $[\mathrm{PGL}_2(\mathbb{Q}_{17}) : H_2] = [H_2 : \mathrm{PSL}_2(\mathbb{Q}_{17})] = 2$.

(4) This follows from Proposition 8.

(5) Let

$$\hat{\psi} : V \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$
$$x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right).$$

It is a group homomorphism such that $\hat{\psi}(x) = \psi(x)$, if $x \in \mathbb{H}(\mathbb{Z}) \cap V$. We have

$$\begin{aligned} \hat{\psi}(V) &= \langle \hat{\psi}(1+2i+2j+2k),\, \hat{\psi}(3+2i),\, \hat{\psi}(1+4j),\, \hat{\psi}(3+2i+2j) \rangle \\ &= \langle \psi(1+2i+2j+2k),\, \psi(3+2i),\, \psi(1+4j),\, \psi(3+2i+2j) \rangle \\ &= \langle a_1, a_5, b_2, b_4 \rangle < \Gamma. \end{aligned}$$

In fact, GAP ([28]) shows that $[\Gamma : \langle a_1, a_5, b_2, b_4 \rangle] = 1$, i.e. $\langle a_1, a_5, b_2, b_4 \rangle = \Gamma$. Therefore $\Gamma = \hat{\psi}(V) \cong V/\ker(\hat{\psi})$. We claim that $\ker(\hat{\psi}) = ZV$. On one hand, we have

$$\ker(\hat{\psi}) = \{x \in V : x = \mathrm{Re}(x)\} = V \cap ZU(\mathbb{H}(\mathbb{Q})) < ZV.$$

On the other hand, if $x = x_0 + x_1 i + x_2 j + x_3 k \in V < U(\mathbb{H}(\mathbb{Q}))$ commutes both with $3 + 2i \in V$ and $1 + 4j \in V$, then $x = \mathrm{Re}(x) \neq 0$, hence $x \in \ker(\hat{\psi})$ and in particular $ZV < \ker(\hat{\psi})$.

□

Note that the only commuting pairs among the generators are $\{a_5, b_1\}$, $\{a_6, b_2\}$ and $\{a_7, b_3\}$.

5.2.2 Example: $p = 5$, $l = 13$

Our second example is $\Gamma = \Gamma_{5,13}$, using the identification

$$\begin{aligned} a_1 &= \psi(1+2i), \\ a_2 &= \psi(1+2j), \\ a_3 &= \psi(1+2k), \end{aligned}$$

$$\begin{aligned} b_1 &= \psi(1+2i+2j+2k), \\ b_2 &= \psi(1+2i+2j-2k), \\ b_3 &= \psi(1+2i-2j+2k), \\ b_4 &= \psi(1-2i+2j+2k), \\ b_5 &= \psi(3+2i), \\ b_6 &= \psi(3+2j), \\ b_7 &= \psi(3+2k). \end{aligned}$$

Example 41.

$$R(3,7) := \left\{ \begin{array}{lll} a_1b_1a_3b_6^{-1}, & a_1b_2a_2b_7, & a_1b_3a_2^{-1}b_7^{-1}, \\ a_1b_4a_1b_1^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_6a_3b_3, \\ a_1b_7a_2^{-1}b_4^{-1}, & a_1b_7^{-1}a_2b_1, & a_1b_6^{-1}a_3^{-1}b_2, \\ a_1b_4^{-1}a_3^{-1}b_6, & a_1b_3^{-1}a_1b_2^{-1}, & a_2b_2a_3^{-1}b_5^{-1}, \\ a_2b_3a_2b_1^{-1}, & a_2b_4a_3b_5, & a_2b_5a_3^{-1}b_3^{-1}, \\ a_2b_6a_2^{-1}b_6^{-1}, & a_2b_5^{-1}a_3b_1, & a_2b_4^{-1}a_2b_2^{-1}, \\ a_3b_2a_3b_1^{-1}, & a_3b_7a_3^{-1}b_7^{-1}, & a_3b_4^{-1}a_3b_3^{-1} \end{array} \right\}.$$

Theorem 41. (1) $P_h \cong \mathrm{PGL}_2(5) < S_6$, $P_v \cong \mathrm{PGL}_2(13) < S_{14}$.

(2) $P_h(X_0) \cong \mathrm{PSL}_2(5)$, $P_v(X_0) \cong \mathrm{PSL}_2(13)$, *independent of the four vertices of* X_0.

(3) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^3$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_{16}^3$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(4)

$$\begin{gathered} \Gamma/\langle\langle b_1^3, b_5^2, (a_1a_2)^3, (b_1b_5)^3 \rangle\rangle_\Gamma \cong \mathrm{PGL}_2(3) \cong S_4, \\ \Gamma/\langle\langle a_1^8, (a_1a_2)^3, (a_1b_1)^7, (b_1b_5)^7, (a_1b_1b_5)^6 \rangle\rangle_\Gamma \cong \mathrm{PGL}_2(7), \\ \Gamma/\langle\langle b_1^4, (b_1b_5)^3, (a_1a_2)^5, (a_1b_1b_5)^5 \rangle\rangle_\Gamma \cong \mathrm{PGL}_2(11), \\ \Gamma/\langle\langle b_2^9, b_5^8, (a_1a_2)^9, (a_1a_3)^9, (b_2b_6)^8, (a_1b_1b_5)^2 \rangle\rangle_\Gamma \cong \mathrm{PGL}_2(17), \\ \Gamma/\langle\langle a_1^5, a_2^5, a_3^5, b_5^{20} \rangle\rangle_\Gamma \cong \mathrm{PGL}_2(19), \\ \Gamma/\langle\langle b_4^{12}, b_5^3, b_6^3, (b_4b_5)^{11} \rangle\rangle_\Gamma \cong \mathrm{PGL}_2(23), \\ \Gamma/\langle\langle a_1^{14}, b_1^5, b_5^7, b_6^7, (a_1b_1)^3 \rangle\rangle_\Gamma \cong \mathrm{PSL}_2(29). \end{gathered}$$

(5) *We get a finite presentation of*

$$U(\mathbb{H}(\mathbb{Z}[1/5,1/13]))/ZU(\mathbb{H}(\mathbb{Z}[1/5,1/13]))$$

by adding to the presentation $\langle a_1,a_2,a_3,b_1,\ldots,b_7 \mid R(3,7)\rangle$ *of* Γ *two new generators* i, j *and the relations*

$$\begin{gathered} i^2=1, j^2=1, ij=ji, \\ a_1i=ia_1, a_2i=ia_2^{-1}, a_3i=ia_3^{-1}, \\ a_1j=ja_1^{-1}, a_2j=ja_2, a_3j=ja_3^{-1}, \\ b_1i=ib_4^{-1}, b_2i=ib_3, b_3i=ib_2, b_4i=ib_1^{-1}, b_5i=ib_5, b_6i=ib_6^{-1}, b_7i=ib_7^{-1}, \\ b_1j=jb_3^{-1}, b_2j=jb_4, b_3j=jb_1^{-1}, b_4j=jb_2, b_5j=jb_5^{-1}, b_6j=jb_6, b_7j=jb_7^{-1}. \end{gathered}$$

and Γ *is then the kernel of*

$$\begin{aligned} U(\mathbb{H}(\mathbb{Z}[1/5,1/13]))/ZU(\mathbb{H}(\mathbb{Z}[1/5,1/13])) &\to \mathbb{Z}_2^2 \\ i &\mapsto (1,0), \\ j &\mapsto (0,1), \\ a_1,a_2,a_3 &\mapsto (0,0), \\ b_1,\ldots,b_7 &\mapsto (0,0). \end{aligned}$$

(6) $(U(\mathbb{H}(\mathbb{Z}[1/5,1/13]))/ZU(\mathbb{H}(\mathbb{Z}[1/5,1/13])))^{ab} \cong \mathbb{Z}_2^6$.

(7) $\Gamma < \mathrm{SO}_3(\mathbb{Q})$ *(illustrating Proposition 48(2)).*

(8) $Z_\Gamma(b) = N_\Gamma(\langle b\rangle) = \langle b\rangle$, *if* $b \in \{b_1,\ldots,b_4\}$.

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (1,6,3,4,2,5),\\
\rho_v(b_2) &= (1,6,2,5,4,3),\\
\rho_v(b_3) &= (1,6,5,2,3,4),\\
\rho_v(b_4) &= (1,2,5,3,4,6),\\
\rho_v(b_5) &= (2,3,5,4),\\
\rho_v(b_6) &= (1,4,6,3),\\
\rho_v(b_7) &= (1,2,6,5),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,4,7,3,13,9,11,14,8,2,12,6),\\
\rho_h(a_2) &= (1,3,5,2,11,8,12,14,10,4,13,7),\\
\rho_h(a_3) &= (1,2,6,4,12,10,13,14,9,3,11,5).
\end{aligned}$$

(2) GAP ([28]).

(3) GAP ([28]).

(4) GAP ([28]). To illustrate Proposition 48(3) and (4), the (surjective) homomorphism $\tau_{2,3}$: $\Gamma \to \mathrm{PGL}_2(7)$ with kernel $\langle\!\langle a_1^8, (a_1a_2)^3, (a_1b_1)^7, (b_1b_5)^7, (a_1b_1b_5)^6 \rangle\!\rangle_\Gamma$ is given by

$$\begin{aligned}
a_1 &\mapsto \left[\begin{pmatrix} 5+7\mathbb{Z} & 1+7\mathbb{Z} \\ 1+7\mathbb{Z} & 4+7\mathbb{Z} \end{pmatrix}\right],\\
a_2 &\mapsto \left[\begin{pmatrix} 1+7\mathbb{Z} & 2+7\mathbb{Z} \\ 5+7\mathbb{Z} & 1+7\mathbb{Z} \end{pmatrix}\right],\\
a_3 &\mapsto \left[\begin{pmatrix} 0+7\mathbb{Z} & 4+7\mathbb{Z} \\ 4+7\mathbb{Z} & 2+7\mathbb{Z} \end{pmatrix}\right],
\end{aligned}$$

$$\begin{aligned}
b_1 &\mapsto \left[\begin{pmatrix} 4+7\mathbb{Z} & 0+7\mathbb{Z} \\ 3+7\mathbb{Z} & 5+7\mathbb{Z} \end{pmatrix}\right],\\
b_2 &\mapsto \left[\begin{pmatrix} 6+7\mathbb{Z} & 6+7\mathbb{Z} \\ 2+7\mathbb{Z} & 3+7\mathbb{Z} \end{pmatrix}\right],\\
b_3 &\mapsto \left[\begin{pmatrix} 4+7\mathbb{Z} & 3+7\mathbb{Z} \\ 0+7\mathbb{Z} & 5+7\mathbb{Z} \end{pmatrix}\right],\\
b_4 &\mapsto \left[\begin{pmatrix} 3+7\mathbb{Z} & 5+7\mathbb{Z} \\ 1+7\mathbb{Z} & 6+7\mathbb{Z} \end{pmatrix}\right],\\
b_5 &\mapsto \left[\begin{pmatrix} 0+7\mathbb{Z} & 1+7\mathbb{Z} \\ 1+7\mathbb{Z} & 6+7\mathbb{Z} \end{pmatrix}\right],\\
b_6 &\mapsto \left[\begin{pmatrix} 3+7\mathbb{Z} & 2+7\mathbb{Z} \\ 5+7\mathbb{Z} & 3+7\mathbb{Z} \end{pmatrix}\right],\\
b_7 &\mapsto \left[\begin{pmatrix} 2+7\mathbb{Z} & 4+7\mathbb{Z} \\ 4+7\mathbb{Z} & 4+7\mathbb{Z} \end{pmatrix}\right].
\end{aligned}$$

This corresponds to the permutation representation in S_8 found by quotpic ([59]):

$$\begin{aligned}
a_1 &\mapsto (1,5,7,2,4,6,3,8),\\
a_2 &\mapsto (1,5,6,4,8,3,7,2),\\
a_3 &\mapsto (1,5,3,8,2,7,6,4),\\
\\
b_1 &\mapsto (2,6,4,3,8,7),\\
b_2 &\mapsto (1,5,4,6,8,3),\\
b_3 &\mapsto (1,5,2,7,4,6),\\
b_4 &\mapsto (1,5,8,3,2,7),\\
b_5 &\mapsto (1,6,7,8,4,5,3,2),\\
b_6 &\mapsto (1,3,6,2,8,5,7,4),\\
b_7 &\mapsto (1,7,3,4,2,5,6,8).
\end{aligned}$$

For $q = 29$, we have $\tau_{12,0}(\Gamma) = \mathrm{PSL}_2(29) < \mathrm{PGL}_2(29)$, given by

$$\begin{aligned}
a_1 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 0+29\mathbb{Z} \\ 0+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right],\\
a_2 &\mapsto \left[\begin{pmatrix} 1+29\mathbb{Z} & 2+29\mathbb{Z} \\ 27+29\mathbb{Z} & 1+29\mathbb{Z} \end{pmatrix}\right],\\
a_3 &\mapsto \left[\begin{pmatrix} 1+29\mathbb{Z} & 24+29\mathbb{Z} \\ 24+29\mathbb{Z} & 1+29\mathbb{Z} \end{pmatrix}\right],\\
\\
b_1 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 26+29\mathbb{Z} \\ 22+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right],\\
b_2 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 7+29\mathbb{Z} \\ 3+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right],\\
b_3 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 22+29\mathbb{Z} \\ 26+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right],\\
b_4 &\mapsto \left[\begin{pmatrix} 6+29\mathbb{Z} & 26+29\mathbb{Z} \\ 22+29\mathbb{Z} & 25+29\mathbb{Z} \end{pmatrix}\right],\\
b_5 &\mapsto \left[\begin{pmatrix} 27+29\mathbb{Z} & 0+29\mathbb{Z} \\ 0+29\mathbb{Z} & 8+29\mathbb{Z} \end{pmatrix}\right],\\
b_6 &\mapsto \left[\begin{pmatrix} 3+29\mathbb{Z} & 2+29\mathbb{Z} \\ 27+29\mathbb{Z} & 3+29\mathbb{Z} \end{pmatrix}\right],\\
b_7 &\mapsto \left[\begin{pmatrix} 3+29\mathbb{Z} & 24+29\mathbb{Z} \\ 24+29\mathbb{Z} & 3+29\mathbb{Z} \end{pmatrix}\right].
\end{aligned}$$

and kernel $\langle\langle a_1^{14}, b_1^5, b_5^7, b_6^7, (a_1b_1)^3\rangle\rangle_\Gamma$. The choice $c = 17$, $d = 0$ gives another homomorphism $\tau_{17,0} : \Gamma \twoheadrightarrow \mathrm{PSL}_2(29)$ with $\ker(\tau_{17,0}) = \ker(\tau_{12,0})$. Note that $q = 29$ is the smallest odd prime such that $\left(\frac{5}{q}\right) = \left(\frac{13}{q}\right) = 1$, see Table 15.

(5) This follows from Proposition 48(1). Observe that i and j in the given presentation correspond to

$$\psi(i) = \left(\left[\begin{pmatrix} i_5 & 0 \\ 0 & -i_5 \end{pmatrix}\right], \left[\begin{pmatrix} i_{13} & 0 \\ 0 & -i_{13} \end{pmatrix}\right]\right) \in \mathrm{PGL}_2(\mathbb{Q}_5) \times \mathrm{PGL}_2(\mathbb{Q}_{13})$$

and
$$\psi(j) = \left(\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\right) \in \mathrm{PGL}_2(\mathbb{Q}_5) \times \mathrm{PGL}_2(\mathbb{Q}_{13})$$
respectively. Note that it would be enough to add the relations $i^2 = 1$, $j^2 = 1$, $ij = ji$, $a_1 i = i a_1$, $a_1 j = j a_1^{-1}$, $a_2 j = j a_2$, $a_3 j = j a_3^{-1}$, $b_1 i = i b_4^{-1}$, $b_5 i = i b_5$, $b_6 i = i b_6^{-1}$, $b_1 j = j b_3^{-1}$ in order to get a presentation of $U(\mathbb{H}(\mathbb{Z}[1/5, 1/13]))/ZU(\mathbb{H}(\mathbb{Z}[1/5, 1/13]))$.

(6) This follows from the presentation given in (5).

(7) The injective group homomorphism $\Gamma \to \mathrm{SO}_3(\mathbb{Q})$ of Proposition 48(2) is given by
$$a_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix},$$
$$a_2 \mapsto \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix},$$
$$a_3 \mapsto \begin{pmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$b_1 \mapsto \frac{1}{13}\begin{pmatrix} -3 & 4 & 12 \\ 12 & -3 & 4 \\ 4 & 12 & -3 \end{pmatrix},$$
$$b_2 \mapsto \frac{1}{13}\begin{pmatrix} -3 & 12 & -4 \\ 4 & -3 & -12 \\ -12 & -4 & -3 \end{pmatrix},$$
$$b_3 \mapsto \frac{1}{13}\begin{pmatrix} -3 & -12 & 4 \\ -4 & -3 & -12 \\ 12 & -4 & -3 \end{pmatrix},$$
$$b_4 \mapsto \frac{1}{13}\begin{pmatrix} -3 & -12 & -4 \\ -4 & -3 & 12 \\ -12 & 4 & -3 \end{pmatrix},$$
$$b_5 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/13 & -12/13 \\ 0 & 12/13 & 5/13 \end{pmatrix},$$
$$b_6 \mapsto \begin{pmatrix} 5/13 & 0 & 12/13 \\ 0 & 1 & 0 \\ -12/13 & 0 & 5/13 \end{pmatrix},$$
$$b_7 \mapsto \begin{pmatrix} 5/13 & -12/13 & 0 \\ 12/13 & 5/13 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(8) This follows from Proposition 8.

□

See Table 18 for the index $[\Gamma : U]$ and the abelianization U^{ab}, where $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1, a_2, a_3\}$, $b_j \in \{b_1, \ldots, b_7\}$.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7
a_1	$16, [16, 32]$	$16, [16, 32]$	$16, [16, 32]$	$16, [16, 32]$	$\infty, [0, 0]$	$96, [16, 32]$	$96, [16, 32]$
a_2	$16, [16, 32]$	$16, [16, 32]$	$16, [16, 32]$	$16, [16, 32]$	$96, [16, 32]$	$\infty, [0, 0]$	$96, [16, 32]$
a_3	$16, [16, 32]$	$16, [16, 32]$	$16, [16, 32]$	$16, [16, 32]$	$96, [16, 32]$	$96, [16, 32]$	$\infty, [0, 0]$

Table 18: $[\Gamma : U], U^{ab}$, where $U = \langle a_i, b_j \rangle$ in Example 41

5.3 Generalization to $p, l \equiv 3 \pmod 4$

The goal of this section is to generalize the construction of $\Gamma_{p,l}$ of Section 5.2 to the case where $p \equiv 3 \pmod 4$ and $l \equiv 3 \pmod 4$ are distinct primes. If we just naively define

$$\Gamma := \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\},$$

then we have several problems:

(1) The condition "x has type e_0" is not preserved under quaternion multiplication (for example $(i + j + k)^2 = -3$ has type o_0), so we better define Γ just as group generated by $a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}$, where

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p\},$$

$$\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |y|^2 = l\}$$

or (as will be explained in (3))

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |x|^2 = p\},$$

$$\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |y|^2 = l\},$$

i.e.

$$\begin{aligned}\Gamma &= \{\psi(x) \mid |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ \text{if } |x|^2 \equiv 3 \pmod 4, \\ &\qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \text{if } |x|^2 \equiv 1 \pmod 4\} \\ &= \{\psi(x) \mid |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ \text{if } r + s \text{ is odd}, \\ &\qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \text{if } r + s \text{ is even}\},\end{aligned}$$

or

$$\begin{aligned}\Gamma &= \{\psi(x) \mid |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \text{if } |x|^2 \equiv 3 \pmod 4, \\ &\qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \text{if } |x|^2 \equiv 1 \pmod 4\} \\ &= \{\psi(x) \mid |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \text{if } r + s \text{ is odd}, \\ &\qquad\qquad\qquad\qquad\qquad\quad x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \text{if } r + s \text{ is even}\}\end{aligned}$$

for suitable ψ, see (2).

(2) What is ψ? There are no elements $i_p \in \mathbb{Q}_p$, $i_l \in \mathbb{Q}_l$ anymore such that $i_p^2 + 1 = 0$, $i_l^2 + 1 = 0$. We have two possibilities to generalize ψ: Either we define $\psi : \mathbb{H}(\mathbb{Z}) \to \mathrm{PGL}_2(K_p) \times \mathrm{PGL}_2(K_l)$, $x = x_0 + x_1 i + x_2 j + x_3 k \mapsto$

$$\left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

where K_p, K_l are quadratic extensions of $\mathbb{Q}_p$ and $\mathbb{Q}_l$ respectively, containing elements $i_p \in K_p$, $i_l \in K_l$ such that $i_p^2 + 1 = 0$, $i_l^2 + 1 = 0$,
or we define $\psi : \mathbb{H}(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$, $x = x_0 + x_1 i + x_2 j + x_3 k \mapsto$

$$\left(\left[\begin{pmatrix} x_0 + x_1 c_p + x_3 d_p & -x_1 d_p + x_2 + x_3 c_p \\ -x_1 d_p - x_2 + x_3 c_p & x_0 - x_1 c_p - x_3 d_p \end{pmatrix}\right], \left[\begin{pmatrix} x_0 + x_1 c_l + x_3 d_l & -x_1 d_l + x_2 + x_3 c_l \\ -x_1 d_l - x_2 + x_3 c_l & x_0 - x_1 c_l - x_3 d_l \end{pmatrix}\right]\right),$$

where $c_p, d_p \in \mathbb{Q}_p$, $c_l, d_l \in \mathbb{Q}_l$ are elements such that $c_p^2 + d_p^2 + 1 = 0$, $c_l^2 + d_l^2 + 1 = 0$. Such elements exist since the equation $x^2 + y^2 + 1 = 0$ has solutions in $\mathbb{Z}_p$ and $\mathbb{Z}_l$ (see [22, Proposition 2.4.3]) and then applying Hensel's Lemma. Both constructions of ψ are equivalent in the sense that they will give the same defining relations, hence the same group Γ. This mainly follows from $\psi(xy) = \psi(x)\psi(y)$ for both ψ. Therefore, we will always choose any of those definitions of ψ in the following constructions.

(3) If $p \equiv 3 \pmod 8$, then p can be written as a sum of (0 and) three odd squares (by Lemma 44(2),(3)). So if we take for example one generator $a_1 := \psi(x)$ such that $x = 0 + x_1 i + x_2 j + x_3 k$ and $|x|^2 = x_1^2 + x_2^2 + x_3^2 = p$, then

$$a_1 = \psi(x) = \psi(-x) = \psi(\bar{x}) = \psi(x)^{-1} = a_1^{-1},$$

i.e. $a_1^2 = 1$ in Γ, in particular Γ is not torsion-free and therefore no $(p+1, l+1)$-group. We can avoid this problem by changing the type from e_0 to e_1:

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(x) > 0, |x|^2 = p\}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(y) > 0, |y|^2 = l\}$$

whenever $p \equiv 3 \pmod 8$ or $l \equiv 3 \pmod 8$ (see Section 5.3.3, 5.3.4).

In the remaining case $p, l \equiv 7 \pmod 8$, we essentially (we could replace e_1 by e_2 or e_3) have two possibilities: Either we again take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(x) > 0, |x|^2 = p\}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(y) > 0, |y|^2 = l\}$$

(see Section 5.3.1) or we take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ \mathrm{Re}(x) > 0, |x|^2 = p\}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ \mathrm{Re}(y) > 0, |y|^2 = l\}$$

(see Section 5.3.2). These two constructions give different groups (e.g. we have different abelianizations in our examples, see the list in Section 5.5).

We always avoid type-mixing constructions, since if x has type e_ι, $|x|^2 = p$ and y has type $e_\kappa \neq e_\iota$, $|y|^2 = l$, then $|xy|^2 = pl \equiv 1 \pmod 4$. Hence, by Lemma 44(2), $|xy|^2$ can be written as a sum of three squares (one odd and two even squares). By the following multiplication table (Table 19), xy has type o_1, o_2 or o_3, in particular $\mathrm{Re}(xy)$ is even, so it can happen that $\mathrm{Re}(xy) = 0$, but then $xy = -\overline{xy}$, hence $(xy)^2 = xy(-\overline{xy}) \in \mathbb{Z}$ and $(\psi(xy))^2$ is the identity in Γ which implies that Γ is not torsion-free.

In all constructions of Γ, we have

$$\Gamma_0 = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r} l^{2s}; r, s \in \mathbb{N}_0\} < \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l)$$

as in Section 5.2.

$\cdot$	type o_0	type o_1	type o_2	type o_3	type e_0	type e_1	type e_2	type e_3
type o_0	type o_0	type o_1	type o_2	type o_3	type e_0	type e_1	type e_2	type e_3
type o_1	type o_1	type o_0	type o_3	type o_2	type e_1	type e_0	type e_3	type e_2
type o_2	type o_2	type o_3	type o_0	type o_1	type e_2	type e_3	type e_0	type e_1
type o_3	type o_3	type o_2	type o_1	type o_0	type e_3	type e_2	type e_1	type e_0
type e_0	type e_0	type e_1	type e_2	type e_3	type o_0	type o_1	type o_2	type o_3
type e_1	type e_1	type e_0	type e_3	type e_2	type o_1	type o_0	type o_3	type o_2
type e_2	type e_2	type e_3	type e_0	type e_1	type o_2	type o_3	type o_0	type o_1
type e_3	type e_3	type e_2	type e_1	type e_0	type o_3	type o_2	type o_1	type o_0

Table 19: Multiplication table of quaternion types

5.3.1 $p, l \equiv 7 \pmod 8$, type e_1

Let $p, l \equiv 7 \pmod 8$ be distinct primes. We take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\} = \{\psi(x) \mid x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ x_0 > 0, x_1 > 0, |x|^2 = p\}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\} = \{\psi(y) \mid y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ y_0 > 0, y_1 > 0, |y|^2 = l\}$$

and define $\Gamma_{p,l}$ as group generated by $a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}$. As an illustration of this construction, we give two examples: First $(p, l) = (7, 23)$.

$$\begin{aligned}
a_1 &= \psi(1 + 2i + j + k), \\
a_2 &= \psi(1 + 2i + j - k), \\
a_3 &= \psi(1 + 2i - j + k), \\
a_4 &= \psi(1 + 2i - j - k), \\
\\
b_1 &= \psi(1 + 2i + 3j + 3k), \\
b_2 &= \psi(1 + 2i + 3j - 3k), \\
b_3 &= \psi(1 + 2i - 3j - 3k), \\
b_4 &= \psi(1 + 2i - 3j + 3k), \\
b_5 &= \psi(3 + 2i + j + 3k), \\
b_6 &= \psi(3 + 2i + j - 3k), \\
b_7 &= \psi(3 + 2i - j + 3k), \\
b_8 &= \psi(3 + 2i - j - 3k), \\
b_9 &= \psi(3 + 2i + 3j + k), \\
b_{10} &= \psi(3 + 2i - 3j + k), \\
b_{11} &= \psi(3 + 2i + 3j - k), \\
b_{12} &= \psi(3 + 2i - 3j - k)
\end{aligned}$$

defines the $(8, 24)$-group

Example 42.

$$R(4,12) := \left\{ \begin{array}{llll} a_1b_1a_3^{-1}b_4^{-1}, & a_1b_2a_4^{-1}b_5, & a_1b_3a_2b_8, & a_1b_4a_2b_7, \\ a_1b_5a_3^{-1}b_7^{-1}, & a_1b_6a_2^{-1}b_5^{-1}, & a_1b_7a_4^{-1}b_{10}^{-1}, & a_1b_8a_1^{-1}b_{12}, \\ a_1b_9a_4^{-1}b_4, & a_1b_{10}a_3^{-1}b_9^{-1}, & a_1b_{11}a_3b_2, & a_1b_{12}a_3b_3, \\ a_1b_{12}^{-1}a_4^{-1}b_2^{-1}, & a_1b_{11}^{-1}a_2^{-1}b_9, & a_1b_{10}^{-1}a_4b_{11}^{-1}, & a_1b_9^{-1}a_3^{-1}b_{10}, \\ a_1b_7^{-1}a_4b_6^{-1}, & a_1b_6^{-1}a_4^{-1}b_{11}, & a_1b_5^{-1}a_2^{-1}b_6, & a_1b_4^{-1}a_4^{-1}b_8^{-1}, \\ a_1b_3^{-1}a_4b_1^{-1}, & a_1b_2^{-1}a_2^{-1}b_1, & a_1b_1^{-1}a_4b_3^{-1}, & a_2b_1a_4b_9, \\ a_2b_3a_3^{-1}b_{11}, & a_2b_4a_4b_{10}, & a_2b_6a_3^{-1}b_1, & a_2b_9a_3^{-1}b_5^{-1}, \\ a_2b_{10}a_2^{-1}b_7, & a_2b_{12}a_4^{-1}b_{11}^{-1}, & a_2b_{12}^{-1}a_3^{-1}b_8, & a_2b_{11}^{-1}a_4^{-1}b_{12}, \\ a_2b_9^{-1}a_3b_{12}^{-1}, & a_2b_8^{-1}a_4^{-1}b_6, & a_2b_7^{-1}a_3^{-1}b_3^{-1}, & a_2b_5^{-1}a_3b_8^{-1}, \\ a_2b_4^{-1}a_3b_2^{-1}, & a_2b_3^{-1}a_4^{-1}b_2, & a_2b_2^{-1}a_3b_4^{-1}, & a_2b_1^{-1}a_3^{-1}b_{10}^{-1}, \\ a_3b_4a_4^{-1}b_3^{-1}, & a_3b_5a_4b_1, & a_3b_6a_4b_2, & a_3b_8a_4^{-1}b_7^{-1}, \\ a_3b_{10}a_4^{-1}b_{12}^{-1}, & a_3b_{11}a_3^{-1}b_6, & a_3b_7^{-1}a_4^{-1}b_8, & a_4b_5a_4^{-1}b_9 \end{array} \right\}.$$

Theorem 42. *(1) $P_h \cong \mathrm{PSL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(23) < S_{24}$.*

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (1,5,3,2)(4,8,6,7), \\
\rho_v(b_2) &= (1,4,2,6)(3,7,8,5), \\
\rho_v(b_3) &= (1,5,7,6)(2,3,4,8), \\
\rho_v(b_4) &= (1,3,7,4)(2,6,5,8), \\
\rho_v(b_5) &= (1,2,3,7,8,6,4), \\
\rho_v(b_6) &= (1,5,8,7,6,4,2), \\
\rho_v(b_7) &= (1,3,4,8,5,6,7), \\
\rho_v(b_8) &= (1,4,3,2,6,5,7), \\
\rho_v(b_9) &= (1,3,7,6,8,5,2), \\
\rho_v(b_{10}) &= (1,4,8,6,5,2,3), \\
\rho_v(b_{11}) &= (1,5,7,8,3,2,4), \\
\rho_v(b_{12}) &= (2,6,7,5,8,3,4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,22,12,17,3,24,23,11,15,16,14,19,20,2,13,8,21,9,10,7,5,6,18,4), \\
\rho_h(a_2) &= (1,2,21,7,15,4,23,22,8,20,19,17,13,14,3,18,10,24,6,5,9,11,12,16), \\
\rho_h(a_3) &= (1,19,11,22,7,8,12,10,9,13,3,4,23,6,14,2,21,24,5,17,18,20,16,15), \\
\rho_h(a_4) &= (1,22,21,10,14,13,15,18,17,4,16,5,23,12,11,6,8,7,19,2,3,24,9,20).
\end{aligned}$$

(2) GAP ([28]).

$\square$

Our second example is $\Gamma_{7,31}$:

$$\begin{aligned}
a_1 &= \psi(1+2i+j+k),\\
a_2 &= \psi(1+2i+j-k),\\
a_3 &= \psi(1+2i-j+k),\\
a_4 &= \psi(1+2i-j-k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1+2i+j+5k),\\
b_2 &= \psi(1+2i+j-5k),\\
b_3 &= \psi(1+2i-j+5k),\\
b_4 &= \psi(1+2i-j-5k),\\
b_5 &= \psi(1+2i+5j+k),\\
b_6 &= \psi(1+2i+5j-k),\\
b_7 &= \psi(1+2i-5j+k),\\
b_8 &= \psi(1+2i-5j-k),\\
b_9 &= \psi(5+2i+j+k),\\
b_{10} &= \psi(5+2i+j-k),\\
b_{11} &= \psi(5+2i-j+k),\\
b_{12} &= \psi(5+2i-j-k),\\
b_{13} &= \psi(3+2i+3j+3k),\\
b_{14} &= \psi(3+2i+3j-3k),\\
b_{15} &= \psi(3+2i-3j+3k),\\
b_{16} &= \psi(3+2i-3j-3k).
\end{aligned}$$

Example 43.

$$R(4,16) := \left\{ \begin{array}{llll}
a_1b_1a_4^{-1}b_8^{-1}, & a_1b_2a_3^{-1}b_{16}^{-1}, & a_1b_3a_1b_{14}^{-1}, & a_1b_4a_4b_1, \\
a_1b_5a_4b_8, & a_1b_6a_1b_{15}^{-1}, & a_1b_7a_4^{-1}b_{10}^{-1}, & a_1b_8a_3^{-1}b_6^{-1}, \\
a_1b_9a_1^{-1}b_9^{-1}, & a_1b_{10}a_4^{-1}b_3^{-1}, & a_1b_{11}a_4b_{14}, & a_1b_{12}a_2^{-1}b_{11}^{-1}, \\
a_1b_{13}a_1b_{12}^{-1}, & a_1b_{14}a_3^{-1}b_4^{-1}, & a_1b_{15}a_4b_{10}, & a_1b_{16}a_4^{-1}b_{13}^{-1}, \\
a_1b_{16}^{-1}a_2^{-1}b_7, & a_1b_{13}^{-1}a_4^{-1}b_{16}, & a_1b_{11}^{-1}a_4^{-1}b_2, & a_1b_{10}^{-1}a_3^{-1}b_{12}, \\
a_1b_8^{-1}a_2^{-1}b_{15}, & a_1b_7^{-1}a_3b_5^{-1}, & a_1b_6^{-1}a_4^{-1}b_{11}, & a_1b_5^{-1}a_3b_7^{-1}, \\
a_1b_4^{-1}a_4^{-1}b_5, & a_1b_3^{-1}a_2^{-1}b_4, & a_1b_2^{-1}a_2b_1^{-1}, & a_1b_1^{-1}a_2b_2^{-1}, \\
a_2b_1a_3^{-1}b_{12}^{-1}, & a_2b_2a_3b_3, & a_2b_4a_2b_{13}^{-1}, & a_2b_5a_2b_{16}^{-1}, \\
a_2b_6a_3^{-1}b_3^{-1}, & a_2b_7a_3b_6, & a_2b_9a_3b_{16}, & a_2b_{10}a_2^{-1}b_{10}^{-1}, \\
a_2b_{11}a_4^{-1}b_9^{-1}, & a_2b_{12}a_3^{-1}b_5^{-1}, & a_2b_{13}a_3b_{12}, & a_2b_{14}a_2b_{11}^{-1}, \\
a_2b_{15}a_3^{-1}b_{14}^{-1}, & a_2b_{15}^{-1}a_4^{-1}b_1, & a_2b_{14}^{-1}a_3^{-1}b_{15}, & a_2b_9^{-1}a_3^{-1}b_8, \\
a_2b_8^{-1}a_4b_6^{-1}, & a_2b_7^{-1}a_3^{-1}b_2, & a_2b_6^{-1}a_4b_8^{-1}, & a_2b_5^{-1}a_4^{-1}b_7, \\
a_2b_4^{-1}a_3^{-1}b_9, & a_2b_3^{-1}a_4^{-1}b_{13}, & a_3b_1a_3b_{16}^{-1}, & a_3b_2a_4^{-1}b_1^{-1}, \\
a_3b_5a_4^{-1}b_{14}^{-1}, & a_3b_8a_3b_{13}^{-1}, & a_3b_{11}a_3^{-1}b_{11}^{-1}, & a_3b_{13}a_4^{-1}b_6^{-1}, \\
a_3b_{15}a_3b_{10}^{-1}, & a_3b_9^{-1}a_4^{-1}b_{10}, & a_3b_4^{-1}a_4b_3^{-1}, & a_3b_3^{-1}a_4b_4^{-1}, \\
a_4b_2a_4b_{15}^{-1}, & a_4b_7a_4b_{14}^{-1}, & a_4b_{12}a_4^{-1}b_{12}^{-1}, & a_4b_{16}a_4b_9^{-1}]
\end{array} \right\}.$$

Theorem 43. (1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PSL}_2(31) < S_{32}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1)

$$
\begin{aligned}
\rho_v(b_1) &= (1,7,6,3,4,2,8,5),\\
\rho_v(b_2) &= (1,7,3,2,8,6,5,4),\\
\rho_v(b_3) &= (1,4,6,2,3,5,7,8),\\
\rho_v(b_4) &= (1,3,5,8,4,6,7,2),\\
\rho_v(b_5) &= (1,6,5,7,2,3,8,4),\\
\rho_v(b_6) &= (1,3,4,7,6,2,5,8),\\
\rho_v(b_7) &= (1,6,7,3,8,5,4,2),\\
\rho_v(b_8) &= (1,4,7,8,6,3,2,5),\\
\rho_v(b_9) &= (2,4,5,6,7,3),\\
\rho_v(b_{10}) &= (1,4,3,6,8,5),\\
\rho_v(b_{11}) &= (1,2,7,5,8,4),\\
\rho_v(b_{12}) &= (1,8,7,6,2,3),\\
\rho_v(b_{13}) &= (1,4,2,7,3,6,5,8),\\
\rho_v(b_{14}) &= (1,8,6,7,2,3,4,5),\\
\rho_v(b_{15}) &= (1,8,4,5,7,6,3,2),\\
\rho_v(b_{16}) &= (1,3,6,2,7,8,5,4),
\end{aligned}
$$

$$
\begin{aligned}
\rho_h(a_1) = &(1,31,22,27,18,25,5,26,17,20,21,23,15,6,8)\\
&(2,32,4,14,3,10,7,28,29,30,19,11,12,13,16),\\
\rho_h(a_2) = &(1,31,26,28,17,9,11,14,15,8,27,7,16,5,12)\\
&(2,32,18,19,22,21,13,4,3,6,25,24,29,20,30),\\
\rho_h(a_3) = &(1,2,7,28,21,32,17,31,3,29,19,18,23,24,16)\\
&(4,30,27,25,20,12,10,15,14,5,26,6,13,8,9),\\
\rho_h(a_4) = &(1,15,2,11,6,25,32,31,18,10,9,16,13,3,29)\\
&(4,30,23,26,19,28,8,27,20,17,24,22,14,7,5).
\end{aligned}
$$

(2) GAP ([28]).

□

5.3.2 $p, l \equiv 7 \pmod 8$, type e_0

Again, let $p, l \equiv 7 \pmod 8$ be distinct primes, but now we take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\, \mathrm{Re}(x) > 0, |x|^2 = p\}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\, \mathrm{Re}(y) > 0, |y|^2 = l\}.$$

We denote the group

$$\langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \rangle$$

by Γ_{p,l,e_0} to distinguish it from $\Gamma_{p,l}$ defined in Section 5.3.1.

For $p = 7$, $l = 23$, we choose now

$$\begin{aligned}
a_1 &= \psi(2 + i + j + k), \\
a_2 &= \psi(2 + i + j - k), \\
a_3 &= \psi(2 + i - j + k), \\
a_4 &= \psi(2 - i + j + k), \\
\\
b_1 &= \psi(2 + i + 3j + 3k), \\
b_2 &= \psi(2 + i + 3j - 3k), \\
b_3 &= \psi(2 + i - 3j - 3k), \\
b_4 &= \psi(2 + i - 3j + 3k), \\
b_5 &= \psi(2 + 3i + j + 3k), \\
b_6 &= \psi(2 + 3i + j - 3k), \\
b_7 &= \psi(2 - 3i + j - 3k), \\
b_8 &= \psi(2 - 3i + j + 3k), \\
b_9 &= \psi(2 + 3i + 3j + k), \\
b_{10} &= \psi(2 + 3i - 3j + k), \\
b_{11} &= \psi(2 - 3i - 3j + k), \\
b_{12} &= \psi(2 - 3i + 3j + k).
\end{aligned}$$

This defines the $(8, 24)$-group $\Gamma_{7,23,e_0}$

Example 44.

$$R(4,12) := \left\{ \begin{array}{llll}
a_1b_1a_3b_9, & a_1b_2a_1^{-1}b_{12}^{-1}, & a_1b_3a_3^{-1}b_2^{-1}, & a_1b_4a_3b_{10}, \\
a_1b_5a_2b_1, & a_1b_6a_2b_2, & a_1b_7a_2^{-1}b_8^{-1}, & a_1b_8a_1^{-1}b_4^{-1}, \\
a_1b_9a_4b_5, & a_1b_{10}a_1^{-1}b_6^{-1}, & a_1b_{11}a_4^{-1}b_{10}^{-1}, & a_1b_{12}a_4b_8, \\
a_1b_{12}^{-1}a_3^{-1}b_{11}, & a_1b_{11}^{-1}a_2^{-1}b_9^{-1}, & a_1b_9^{-1}a_2^{-1}b_{11}^{-1}, & a_1b_7^{-1}a_3^{-1}b_5^{-1}, \\
a_1b_6^{-1}a_4^{-1}b_7, & a_1b_5^{-1}a_3^{-1}b_7^{-1}, & a_1b_4^{-1}a_2^{-1}b_3, & a_1b_3^{-1}a_4^{-1}b_1^{-1}, \\
a_1b_1^{-1}a_4^{-1}b_3^{-1}, & a_2b_3a_2^{-1}b_7^{-1}, & a_2b_5a_2^{-1}b_{12}, & a_2b_6a_3^{-1}b_{11}^{-1}, \\
a_2b_7a_3^{-1}b_{10}^{-1}, & a_2b_8a_3b_5^{-1}, & a_2b_{10}a_2^{-1}b_1, & a_2b_{12}a_3b_9^{-1}, \\
a_2b_{12}^{-1}a_4^{-1}b_3, & a_2b_{11}^{-1}a_4^{-1}b_2, & a_2b_9^{-1}a_4b_{10}, & a_2b_8^{-1}a_4b_6^{-1}, \\
a_2b_6^{-1}a_4b_8^{-1}, & a_2b_4^{-1}a_3b_2^{-1}, & a_2b_2^{-1}a_3b_4^{-1}, & a_2b_1^{-1}a_4b_4, \\
a_3b_1a_3^{-1}b_6, & a_3b_2a_4b_1^{-1}, & a_3b_3a_4^{-1}b_8^{-1}, & a_3b_4a_4^{-1}b_7^{-1}, \\
a_3b_6a_4b_5^{-1}, & a_3b_8a_3^{-1}b_9, & a_3b_{11}a_3^{-1}b_3^{-1}, & a_3b_{12}^{-1}a_4b_{10}^{-1}, \\
a_3b_{10}^{-1}a_4b_{12}^{-1}, & a_4b_2a_4^{-1}b_5, & a_4b_7a_4^{-1}b_{11}^{-1}, & a_4b_9a_4^{-1}b_4
\end{array} \right\}.$$

Theorem 44. (1) $P_h \cong \mathrm{PSL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(23) < S_{24}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (1,4,7)(3,5,8),\\
\rho_v(b_2) &= (1,3,7)(2,6,4),\\
\rho_v(b_3) &= (1,4,2)(5,8,6),\\
\rho_v(b_4) &= (2,6,5)(3,7,8),\\
\rho_v(b_5) &= (1,3,5)(2,6,8),\\
\rho_v(b_6) &= (2,5,8)(4,7,6),\\
\rho_v(b_7) &= (1,3,4)(6,8,7),\\
\rho_v(b_8) &= (1,2,5)(3,4,7),\\
\rho_v(b_9) &= (1,2,6)(4,7,8),\\
\rho_v(b_{10}) &= (1,4,6)(2,3,5),\\
\rho_v(b_{11}) &= (1,2,3)(5,7,8),\\
\rho_v(b_{12}) &= (3,5,7)(4,6,8),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,22,21,17,12,2,3,24,5,18,19,15,4,8,7,20,9,14,13,23,6,10,11,16),\\
\rho_h(a_2) &= (1,15,16,11,6,17,18,22,13,5,8,19,2,21,24,10,7,3,4,23,14,9,12,20),\\
\rho_h(a_3) &= (1,2,21,10,13,16,8,3,11,12,15,18,5,6,24,9,17,20,7,4,23,22,14,19),\\
\rho_h(a_4) &= (1,4,16,5,23,24,3,12,15,14,18,21,9,10,13,8,19,20,2,11,7,6,17,22).
\end{aligned}$$

Note that any two of the permutations in the set $\{\rho_h(a_1), \rho_h(a_2), \rho_h(a_3), \rho_h(a_4)\}$ already generate $P_v \cong \mathrm{PGL}_2(23)$.

(2) GAP ([28]).

□

5.3.3 $p, l \equiv 3 \pmod 8$

Let $p, l \equiv 3 \pmod 8$ be distinct primes,

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(x) > 0, |x|^2 = p\},$$

$$\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(y) > 0, |y|^2 = l\}$$

and

$$\Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \rangle.$$

We give the example $\Gamma_{3,11}$:

$$\begin{aligned}
a_1 &= \psi(1+j+k),\\
a_2 &= \psi(1+j-k),
\end{aligned}$$

$$
\begin{aligned}
b_1 &= \psi(1+j+3k),\\
b_2 &= \psi(1+j-3k),\\
b_3 &= \psi(1+3j+k),\\
b_4 &= \psi(1+3j-k),\\
b_5 &= \psi(3+j+k),\\
b_6 &= \psi(3+j-k).
\end{aligned}
$$

Example 45.

$$
R(2,6) := \left\{ \begin{array}{ll}
a_1b_1a_1b_6^{-1}, & a_1b_2a_1b_4^{-1},\\
a_1b_3a_1b_6, & a_1b_4a_2^{-1}b_3^{-1},\\
a_1b_5a_1^{-1}b_5^{-1}, & a_1b_3^{-1}a_2^{-1}b_4,\\
a_1b_2^{-1}a_2b_1^{-1}, & a_1b_1^{-1}a_2b_2^{-1},\\
a_2b_1a_2b_3^{-1}, & a_2b_2a_2b_5^{-1},\\
a_2b_4a_2b_5, & a_2b_6a_2^{-1}b_6^{-1}
\end{array} \right\}.
$$

Theorem 45. (1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v \cong \mathrm{PSL}_2(11) < S_{12}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_8^2 \times \mathbb{Z}_{64}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.

Proof. (1)

$$
\begin{aligned}
\rho_v(b_1) &= (1,3,2,4),\\
\rho_v(b_2) &= (1,3,2,4),\\
\rho_v(b_3) &= (1,2,3,4),\\
\rho_v(b_4) &= (1,4,3,2),\\
\rho_v(b_5) &= (2,3),\\
\rho_v(b_6) &= (1,4),
\end{aligned}
$$

$$
\begin{aligned}
\rho_h(a_1) &= (1,11,9,10,6)(2,12,7,3,4),\\
\rho_h(a_2) &= (1,11,8,4,3)(2,12,10,9,5).
\end{aligned}
$$

(2) GAP ([28]).

□

See Table 20 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient G/U (if U is normal in G), where $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1, a_2\}$, $b_j \in \{b_1, \ldots, b_6\}$.

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	$2, [8,8], \mathbb{Z}_2$	$8, [8,32], -$	$2, [8,8], \mathbb{Z}_2$	$8, [8,32], -$	$\infty, [0,0], -$	$2, [8,8], \mathbb{Z}_2$
a_2	$8, [8,32], -$	$2, [8,8], \mathbb{Z}_2$	$8, [8,32], -$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$	$\infty, [0,0], -$

Table 20: $[\Gamma : U], U^{ab}, G/U$, where $U = \langle a_i, b_j \rangle$ in Example 45

5.3.4 $p \equiv 3 \pmod 8$, $l \equiv 7 \pmod 8$

Let $p \equiv 3 \pmod 8$, $l \equiv 7 \pmod 8$ be primes,

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(x) > 0, |x|^2 = p\},$$

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\} = \{\psi(y) \mid y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ y_0 > 0, y_1 > 0, |y|^2 = l\}$$

and

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \rangle.$$

We construct the group $\Gamma_{3,7}$:

$$\begin{aligned} a_1 &= \psi(1 + j + k), \\ a_2 &= \psi(1 + j - k), \end{aligned}$$

$$\begin{aligned} b_1 &= \psi(1 + 2i + j + k), \\ b_2 &= \psi(1 + 2i + j - k), \\ b_3 &= \psi(1 + 2i - j + k), \\ b_4 &= \psi(1 + 2i - j - k). \end{aligned}$$

Example 46.

$$R(2,4) := \left\{ \begin{array}{ll} a_1 b_1 a_2^{-1} b_2^{-1}, & a_1 b_2 a_1^{-1} b_3, \\ a_1 b_3 a_2^{-1} b_4^{-1}, & a_1 b_4 a_1 b_1^{-1}, \\ a_1 b_4^{-1} a_2 b_2, & a_1 b_3^{-1} a_2 b_1, \\ a_2 b_3 a_2 b_2^{-1}, & a_2 b_4 a_2^{-1} b_1 \end{array} \right\}.$$

Theorem 46. (1) $P_h \cong \mathrm{PSL}_2(3) \cong A_4$, $P_v \cong \mathrm{PGL}_2(7) < S_8$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.

(3a)

$$\begin{aligned} \Gamma / \langle\!\langle a_1^6, b_1^4, (a_1 b_1)^5, (b_1 b_2)^5 \rangle\!\rangle_\Gamma &\cong \mathrm{PGL}_2(5) \cong S_5, \\ \Gamma / \langle\!\langle a_1^5, (a_1 b_1)^{12}, (b_1 b_2)^5 \rangle\!\rangle_\Gamma &\cong \mathrm{PGL}_2(11), \\ \Gamma / \langle\!\langle a_1^7, (a_1 b_1)^{14}, (b_1 b_2)^3 \rangle\!\rangle_\Gamma &\cong \mathrm{PGL}_2(13). \end{aligned}$$

(3b)

$$\langle\!\langle a_1^6, b_1^4, (a_1 b_1)^5, (b_1 b_2)^5 \rangle\!\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}^3.$$

(4) $U(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))$ *has a presentation with generators* $a_1, a_2, b_1, b_2, b_3, b_4, i, j$ *and relators*

$$R(2,4),\ a_1 i a_1 i^{-1},\ a_1 j a_2^{-1} j^{-1},\ b_1 i b_4^{-1} i^{-1},\ b_1 j b_3 j^{-1},\ i^2,\ j^2,\ iji^{-1}j^{-1}.$$

(5) $(U(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/7])))^{ab} \cong \mathbb{Z}_2^4$.

(6) $\mathrm{Aut}(X) \cong D_4$.

(7) $\langle a_2^2 a_1^2, b_2^{-1} b_3 b_4 b_1^{-1} \rangle \cong \mathbb{Z}^2$.

Proof. (1)

$$\begin{aligned}\rho_v(b_1) &= (1,4,3),\\ \rho_v(b_2) &= (1,2,3),\\ \rho_v(b_3) &= (2,4,3),\\ \rho_v(b_4) &= (1,2,4),\end{aligned}$$

$$\begin{aligned}\rho_h(a_1) &= (1,4,3,7,5,8,6,2),\\ \rho_h(a_2) &= (1,5,6,7,8,4,2,3).\end{aligned}$$

(2) GAP ([28]).

(3a) Let q be an odd prime distinct from p and l, and choose $c, d \in \mathbb{Z}$ such that $c^2 + d^2 + 1 \equiv 0 \pmod{q}$, then we can define exactly as described in Proposition 48(3) a homomorphism $\tau = \tau_{c,d} : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ by

$$\tau_{c,d}(\gamma) = \left[\begin{pmatrix} x_0 + x_1 c + x_3 d + q\mathbb{Z} & -x_1 d + x_2 + x_3 c + q\mathbb{Z} \\ -x_1 d - x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c - x_3 d + q\mathbb{Z} \end{pmatrix}\right].$$

For $q = 5$ we have $\tau_{0,2} : \Gamma_{3,7} \twoheadrightarrow \mathrm{PGL}_2(5)$ given by

$$\begin{aligned} a_1 &\mapsto \left[\begin{pmatrix} 3+5\mathbb{Z} & 1+5\mathbb{Z} \\ 4+5\mathbb{Z} & 4+5\mathbb{Z} \end{pmatrix}\right], \\ a_2 &\mapsto \left[\begin{pmatrix} 4+5\mathbb{Z} & 1+5\mathbb{Z} \\ 4+5\mathbb{Z} & 3+5\mathbb{Z} \end{pmatrix}\right], \end{aligned}$$

$$\begin{aligned} b_1 &\mapsto \left[\begin{pmatrix} 3+5\mathbb{Z} & 2+5\mathbb{Z} \\ 0+5\mathbb{Z} & 4+5\mathbb{Z} \end{pmatrix}\right], \\ b_2 &\mapsto \left[\begin{pmatrix} 4+5\mathbb{Z} & 2+5\mathbb{Z} \\ 0+5\mathbb{Z} & 3+5\mathbb{Z} \end{pmatrix}\right], \\ b_3 &\mapsto \left[\begin{pmatrix} 3+5\mathbb{Z} & 0+5\mathbb{Z} \\ 2+5\mathbb{Z} & 4+5\mathbb{Z} \end{pmatrix}\right], \\ b_4 &\mapsto \left[\begin{pmatrix} 4+5\mathbb{Z} & 0+5\mathbb{Z} \\ 2+5\mathbb{Z} & 3+5\mathbb{Z} \end{pmatrix}\right]. \end{aligned}$$

In the same way $\tau_{1,3} : \Gamma_{3,7} \twoheadrightarrow \mathrm{PGL}_2(11)$ is defined by

$$\begin{aligned} a_1 &\mapsto \left[\begin{pmatrix} 4+11\mathbb{Z} & 2+11\mathbb{Z} \\ 0+11\mathbb{Z} & 9+11\mathbb{Z} \end{pmatrix}\right], \\ a_2 &\mapsto \left[\begin{pmatrix} 9+11\mathbb{Z} & 0+11\mathbb{Z} \\ 9+11\mathbb{Z} & 4+11\mathbb{Z} \end{pmatrix}\right], \end{aligned}$$

$$\begin{aligned} b_1 &\mapsto \left[\begin{pmatrix} 6+11\mathbb{Z} & 7+11\mathbb{Z} \\ 5+11\mathbb{Z} & 7+11\mathbb{Z} \end{pmatrix}\right], \\ b_2 &\mapsto \left[\begin{pmatrix} 0+11\mathbb{Z} & 5+11\mathbb{Z} \\ 3+11\mathbb{Z} & 2+11\mathbb{Z} \end{pmatrix}\right], \\ b_3 &\mapsto \left[\begin{pmatrix} 6+11\mathbb{Z} & 5+11\mathbb{Z} \\ 7+11\mathbb{Z} & 7+11\mathbb{Z} \end{pmatrix}\right], \\ b_4 &\mapsto \left[\begin{pmatrix} 0+11\mathbb{Z} & 3+11\mathbb{Z} \\ 5+11\mathbb{Z} & 2+11\mathbb{Z} \end{pmatrix}\right] \end{aligned}$$

and $\tau_{0,5} : \Gamma_{3,7} \twoheadrightarrow \mathrm{PGL}_2(13)$ by

$$a_1 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 1+13\mathbb{Z} \\ 12+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right],$$
$$a_2 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 1+13\mathbb{Z} \\ 12+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right],$$

$$b_1 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 4+13\mathbb{Z} \\ 2+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right],$$
$$b_2 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 4+13\mathbb{Z} \\ 2+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right],$$
$$b_3 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 2+13\mathbb{Z} \\ 4+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right],$$
$$b_4 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 2+13\mathbb{Z} \\ 4+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right].$$

($3b$) quotpic ([59])

(4) Same idea as in Theorem 41(5) using

$$U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))/ZU(\mathbb{H}(\mathbb{Z}[1/p, 1/l])) \cong \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0\}.$$

(5) Follows from (4).

(6) GAP ([28]). $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$(a_1, a_2, b_1, b_2, b_3, b_4) \mapsto (a_1, a_2^{-1}, b_4^{-1}, b_2^{-1}, b_3^{-1}, b_1^{-1}),$$
$$(a_1, a_2, b_1, b_2, b_3, b_4) \mapsto (a_2, a_1^{-1}, b_2, b_4, b_1, b_3).$$

(7) This follows since $a_2^2 a_1^2 = \psi(1 + 8i - 4j)$ and $b_2^{-1} b_3 b_4 b_1^{-1} = \psi(41 - 24i + 12j)$ commute.

□

See Table 21 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient G/U, where $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1, a_2\}$, $b_j \in \{b_1, \dots, b_4\}$.

	b_1	b_2	b_3	b_4
a_1	$4, [8, 16], \mathbb{Z}_4$	$2, [8, 8], \mathbb{Z}_2$	$2, [8, 8], \mathbb{Z}_2$	$4, [8, 16], \mathbb{Z}_4$
a_2	$2, [8, 8], \mathbb{Z}_2$	$4, [8, 16], \mathbb{Z}_4$	$4, [8, 16], \mathbb{Z}_4$	$2, [8, 8], \mathbb{Z}_2$

Table 21: $[\Gamma : U], U^{ab}, G/U$, where $U = \langle a_i, b_j \rangle$ in Example 46

5.4 Mixed examples: $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$

Let $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$ be primes. Similarly as in Section 5.3, we construct groups $\Gamma_{p,l}$ generated by

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \mathrm{Re}(x) > 0, |x|^2 = p\}$$

and

$$\{b_1,\ldots,b_{\frac{l+1}{2}}\}^{\pm 1}=\{\psi(y)\mid y\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\,\mathrm{Re}(y)>0,|y|^2=l\},$$

(see Section 5.4.1, 5.4.3), i.e.

$$\begin{aligned}\Gamma_{p,l}&=\{\psi(x)\mid |x|^2=p^rl^s;r,s\in\mathbb{N}_0,x\in\mathbb{H}(\mathbb{Z})\text{ has type }e_1,\ \text{if }|x|^2\equiv 3\pmod 4,\\&\qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\ \text{if }|x|^2\equiv 1\pmod 4\}\\&=\{\psi(x)\mid |x|^2=p^rl^s;r,s\in\mathbb{N}_0,x\in\mathbb{H}(\mathbb{Z})\text{ has type }e_1,\ \text{if }r\text{ is odd},\\&\qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\ \text{if }r\text{ is even}\},\end{aligned}$$

and groups Γ_{p,l,e_0} generated by

$$\{a_1,\ldots,a_{\frac{p+1}{2}}\}^{\pm 1}=\{\psi(x)\mid x\in\mathbb{H}(\mathbb{Z})\text{ has type }e_0,\,\mathrm{Re}(x)>0,|x|^2=p\equiv 7\,(\mathrm{mod}\ 8)\}$$

and

$$\{b_1,\ldots,b_{\frac{l+1}{2}}\}^{\pm 1}=\{\psi(y)\mid y\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\,\mathrm{Re}(y)>0,|y|^2=l\equiv 1\,(\mathrm{mod}\ 8)\},$$

(see Section 5.4.2), i.e.

$$\begin{aligned}\Gamma_{p,l,e_0}&=\{\psi(x)\mid |x|^2=p^rl^s;r,s\in\mathbb{N}_0,x\in\mathbb{H}(\mathbb{Z})\text{ has type }e_0,\ \text{if }|x|^2\equiv 7\pmod 8,\\&\qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\ \text{if }|x|^2\equiv 1\pmod 8\}\\&=\{\psi(x)\mid |x|^2=p^rl^s;r,s\in\mathbb{N}_0,x\in\mathbb{H}(\mathbb{Z})\text{ has type }e_0,\ \text{if }r\text{ is odd},\\&\qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\ \text{if }r\text{ is even}\}.\end{aligned}$$

Note that for both constructions $\Gamma_{p,l}$ and Γ_{p,l,e_0} we have

$$\Gamma_0=\{\psi(x)\mid x\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\,|x|^2=p^{2r}l^{2s};r,s\in\mathbb{N}_0\}<\mathrm{PSL}_2(\mathbb{Q}_p)\times\mathrm{PSL}_2(\mathbb{Q}_l).$$

as in Section 5.2 and 5.3.

5.4.1 $p\equiv 7\pmod 8$, type e_1

Let $p\equiv 7\pmod 8$, $l\equiv 1\pmod 4$ be primes,

$$\{a_1,\ldots,a_{\frac{p+1}{2}}\}=\{\psi(x)\mid x=x_0+x_1i+x_2j+x_3k\in\mathbb{H}(\mathbb{Z})\text{ has type }e_1,\,x_0>0,x_1>0,|x|^2=p\},$$

$$\{b_1,\ldots,b_{\frac{l+1}{2}}\}^{\pm 1}=\{\psi(y)\mid y\in\mathbb{H}(\mathbb{Z})\text{ has type }o_0,\,\mathrm{Re}(y)>0,|y|^2=l\}$$

and

$$\Gamma_{p,l}=\langle a_1,\ldots,a_{\frac{p+1}{2}},b_1,\ldots,b_{\frac{l+1}{2}}\rangle.$$

We study two examples: $\Gamma_{7,5}$ is given by

$$\begin{aligned}a_1&=\psi(1+2i+j+k),\\a_2&=\psi(1+2i+j-k),\\a_3&=\psi(1+2i-j+k),\\a_4&=\psi(1+2i-j-k),\end{aligned}$$

$$\begin{aligned}b_1&=\psi(1+2i),\\b_2&=\psi(1+2j),\\b_3&=\psi(1+2k).\end{aligned}$$

Example 47.

$$R(4,3) := \left\{ \begin{array}{llll} a_1b_1a_3b_3^{-1}, & a_1b_2a_4b_2^{-1}, & a_1b_3a_4^{-1}b_2, & a_1b_3^{-1}a_4b_3, \\ a_1b_2^{-1}a_2b_1, & a_1b_1^{-1}a_4b_1^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, & a_2b_3a_4b_1, \\ a_2b_3^{-1}a_3b_3, & a_2b_2^{-1}a_3b_2, & a_2b_1^{-1}a_3b_1^{-1}, & a_3b_1a_4b_2 \end{array} \right\}.$$

Theorem 47. (1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(3) $\mathrm{Aut}(X) \cong S_4$.

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (1,5,2,6,4,8,3,7), \\ \rho_v(b_2) &= (1,5,3,7,6,2,8,4), \\ \rho_v(b_3) &= (1,6,2,3,7,4,8,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,6,5,3), \\ \rho_h(a_2) &= (1,6,3,2), \\ \rho_h(a_3) &= (1,6,4,5), \\ \rho_h(a_4) &= (1,6,2,4). \end{aligned}$$

(2) GAP ([28]).

(3) GAP ([28]). $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$\begin{aligned} (a_1,a_2,a_3,a_4,b_1,b_2,b_3) &\mapsto (a_1,a_3,a_4,a_2,b_3,b_1,b_2), \\ (a_1,a_2,a_3,a_4,b_1,b_2,b_3) &\mapsto (a_2,a_4^{-1},a_1,a_3^{-1},b_1,b_3^{-1},b_2^{-1}). \end{aligned}$$

∎

See Table 22 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient G/U, where $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1,\dots,a_4\}$, $b_j \in \{b_1,b_2,b_3\}$.

	b_1	b_2	b_3
a_1	$4, [8,16], \mathbb{Z}_4$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$
a_2	$4, [8,16], \mathbb{Z}_4$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$
a_3	$4, [8,16], \mathbb{Z}_4$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$
a_4	$4, [8,16], \mathbb{Z}_4$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$

Table 22: $[\Gamma : U], U^{ab}, G/U$, where $U = \langle a_i, b_j \rangle$ in Example 47

Our second example is $\Gamma_{7,13}$:

$$\begin{aligned} a_1 &= \psi(1+2i+j+k), \\ a_2 &= \psi(1+2i+j-k), \\ a_3 &= \psi(1+2i-j+k), \\ a_4 &= \psi(1+2i-j-k), \end{aligned}$$

$$
\begin{aligned}
b_1 &= \psi(1+2i+2j+2k),\\
b_2 &= \psi(1+2i+2j-2k),\\
b_3 &= \psi(1+2i-2j+2k),\\
b_4 &= \psi(1-2i+2j+2k),\\
b_5 &= \psi(3+2i),\\
b_6 &= \psi(3+2j),\\
b_7 &= \psi(3+2k).
\end{aligned}
$$

Example 48.

$$
R(4,7) := \left\{
\begin{array}{llll}
a_1b_1a_1b_5^{-1}, & a_1b_2a_4b_3, & a_1b_3a_1^{-1}b_2^{-1}, & a_1b_4a_4b_1^{-1}, \\
a_1b_5a_2b_6, & a_1b_6a_2^{-1}b_3^{-1}, & a_1b_7a_3b_5, & a_1b_7^{-1}a_3^{-1}b_4^{-1}, \\
a_1b_6^{-1}a_4^{-1}b_7^{-1}, & a_1b_4^{-1}a_2^{-1}b_6^{-1}, & a_1b_2^{-1}a_3^{-1}b_7, & a_1b_1^{-1}a_4b_4, \\
a_2b_1a_2^{-1}b_4, & a_2b_2a_2b_5^{-1}, & a_2b_3a_4^{-1}b_7, & a_2b_5a_4b_7^{-1}, \\
a_2b_7a_3^{-1}b_6^{-1}, & a_2b_7^{-1}a_4^{-1}b_1^{-1}, & a_2b_4^{-1}a_3b_1, & a_2b_3^{-1}a_3b_2^{-1}, \\
a_2b_2^{-1}a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_1, & a_3b_6a_4^{-1}b_2, \\
a_3b_6^{-1}a_4b_5, & a_3b_1^{-1}a_4^{-1}b_6^{-1}, & a_4b_2a_4^{-1}b_3^{-1}, & a_4b_5^{-1}a_4b_4^{-1}
\end{array}
\right\}.
$$

Theorem 48. (1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(13) < S_{14}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1)

$$
\begin{aligned}
\rho_v(b_1) &= (1,5,6,2,4,8),\\
\rho_v(b_2) &= (2,6,8,4,3,7),\\
\rho_v(b_3) &= (1,2,6,3,7,5),\\
\rho_v(b_4) &= (1,3,7,8,4,5),\\
\rho_v(b_5) &= (1,8,2,7,4,5,3,6),\\
\rho_v(b_6) &= (1,2,3,4,6,5,8,7),\\
\rho_v(b_7) &= (1,4,2,5,7,6,8,3),
\end{aligned}
$$

$$
\begin{aligned}
\rho_h(a_1) &= (1,4,8,13,12,2,3,6,11,14,10,7,9,5),\\
\rho_h(a_2) &= (1,8,3,13,10,6,7,5,2,12,9,4,14,11),\\
\rho_h(a_3) &= (1,11,7,2,12,10,9,8,5,3,13,6,14,4),\\
\rho_h(a_4) &= (1,4,10,8,6,5,11,14,7,12,13,3,2,9).
\end{aligned}
$$

(2) GAP ([28]).

□

5.4.2 $p \equiv 7 \pmod 8$, type e_0, $l \equiv 1 \pmod 8$

Let $p \equiv 7 \pmod 8$, $l \equiv 1 \pmod 8$ be primes,

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0, \, \mathrm{Re}(x) > 0, |x|^2 = p\}$$

and

$$\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0, \, \mathrm{Re}(y) > 0, |y|^2 = l\}.$$

We denote the group $\langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \rangle$ by Γ_{p,l,e_0}. Note that we have two restrictions in this type e_0 case. Firstly, we exclude $p \equiv 3 \pmod 8$ for the reasons explained in Section 5.3. Secondly, we exclude the case $p \equiv 7 \pmod 8$, $l \equiv 5 \pmod 8$. To motivate it, observe that if x has type e_0, $|x|^2 = p \equiv 7 \pmod 8$ and y has type o_0, $|y|^2 = l \equiv 1 \pmod 8$, then xy has type e_0 such that $|xy|^2 = pl \equiv 7 \pmod 8$, in particular $\mathrm{Re}(xy) \neq 0$ by Lemma 44(2). However, if x has type e_0, $|x|^2 = p \equiv 7 \pmod 8$ and y has type o_0, $|y|^2 = l \equiv 5 \pmod 8$, then xy has type e_0 such that $|xy|^2 = pl \equiv 3 \pmod 8$ and it can happen that $\mathrm{Re}(xy) = 0$. But this means that $xy = -\overline{xy}$, hence $(xy)^2 = xy(-\overline{xy}) \in \mathbb{Z}$. As a consequence, $\psi((xy)^2)$ is the identity in Γ and Γ is therefore not torsion-free (x, y generate a "projective plane"). We will give an example for this phenomenon later in this section, but first we look at the group $\Gamma_{7,17,e_0}$:

$$\begin{aligned}
a_1 &= \psi(2 + i + j + k), \\
a_2 &= \psi(2 + i + j - k), \\
a_3 &= \psi(2 + i - j + k), \\
a_4 &= \psi(2 - i + j + k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1 + 4i), \\
b_2 &= \psi(1 + 4j), \\
b_3 &= \psi(1 + 4k), \\
b_4 &= \psi(3 + 2i + 2j), \\
b_5 &= \psi(3 + 2i - 2j), \\
b_6 &= \psi(3 + 2i + 2k), \\
b_7 &= \psi(3 + 2i - 2k), \\
b_8 &= \psi(3 + 2j + 2k), \\
b_9 &= \psi(3 + 2j - 2k).
\end{aligned}$$

Example 49.

$$R(4,9) := \left\{ \begin{array}{llll} a_1b_1a_2b_4, & a_1b_2a_4b_8, & a_1b_3a_3b_6, & a_1b_4a_2b_2, \\ a_1b_5a_4b_6^{-1}, & a_1b_6a_3b_1, & a_1b_7a_3^{-1}b_2^{-1}, & a_1b_8a_4b_3, \\ a_1b_9a_3b_4^{-1}, & a_1b_9^{-1}a_4^{-1}b_1^{-1}, & a_1b_8^{-1}a_3b_5^{-1}, & a_1b_7^{-1}a_2b_8^{-1}, \\ a_1b_6^{-1}a_2b_9^{-1}, & a_1b_5^{-1}a_2^{-1}b_3^{-1}, & a_1b_4^{-1}a_4b_7, & a_1b_3^{-1}a_2^{-1}b_5, \\ a_1b_2^{-1}a_3^{-1}b_7^{-1}, & a_1b_1^{-1}a_4^{-1}b_9, & a_2b_1a_4^{-1}b_7, & a_2b_6a_3^{-1}b_4^{-1}, \\ a_2b_7a_4^{-1}b_3^{-1}, & a_2b_8a_3b_1^{-1}, & a_2b_9a_3^{-1}b_2, & a_2b_7^{-1}a_3^{-1}b_5, \\ a_2b_6^{-1}a_4b_2^{-1}, & a_2b_5^{-1}a_4^{-1}b_9^{-1}, & a_2b_4^{-1}a_4^{-1}b_8, & a_2b_3^{-1}a_3^{-1}b_9, \\ a_2b_2^{-1}a_4b_6, & a_2b_1^{-1}a_3b_8^{-1}, & a_3b_4a_4b_3^{-1}, & a_3b_5a_4^{-1}b_1, \\ a_3b_8a_4^{-1}b_6^{-1}, & a_3b_9a_4^{-1}b_7^{-1}, & a_3b_3^{-1}a_4b_4^{-1}, & a_3b_2^{-1}a_4^{-1}b_5 \end{array} \right\}.$$

Theorem 49. (1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(17) < S_{18}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1)

$$\begin{aligned} \rho_v(b_1) &= (1,4,3,7,5,8,2,6), \\ \rho_v(b_2) &= (1,3,2,5,6,8,4,7), \\ \rho_v(b_3) &= (1,2,4,6,7,8,3,5), \\ \rho_v(b_4) &= (1,6,4,8,2,3,5,7), \\ \rho_v(b_5) &= (1,6,5,7,8,4,3,2), \\ \rho_v(b_6) &= (1,5,2,8,3,4,7,6), \\ \rho_v(b_7) &= (1,3,4,2,8,6,7,5), \\ \rho_v(b_8) &= (1,7,3,8,4,2,6,5), \\ \rho_v(b_9) &= (1,7,6,5,8,3,2,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,10,18,6,5,11,2,7,17,4,9,13,3,14,16,8,12,15), \\ \rho_h(a_2) &= (1,8,18,4,6,10,16,5,3,7,11,15,2,13,17,9,14,12), \\ \rho_h(a_3) &= (1,11,18,5,7,9,3,4,16,6,8,14,17,12,2,10,15,13), \\ \rho_h(a_4) &= (1,14,13,11,3,15,16,12,10,5,2,6,17,8,4,7,18,9). \end{aligned}$$

(2) GAP ([28]).

□

We illustrate now, why the (type e_0) construction does not work in the case $p \equiv 7 \pmod 8$, $l \equiv 5 \pmod 8$. Take $p = 7$, $l = 5$: if for example $a_1 = \psi(2+i+j+k)$, $b_1 = \psi(1+2i)$, then $\psi((a_1b_1)^2) = \psi(-35) = 1_\Gamma$, i.e. we have a projective plane, Γ is not torsion-free and therefore no

(8, 6)-group. Nevertheless, we can do some computations: If we take

$$\begin{aligned} a_1 &= \psi(2+i+j+k), \\ a_2 &= \psi(2+i+j-k), \\ a_3 &= \psi(2+i-j+k), \\ a_4 &= \psi(2-i+j+k), \end{aligned}$$

$$\begin{aligned} b_1 &= \psi(1+2i), \\ b_2 &= \psi(1+2j), \\ b_3 &= \psi(1+2k), \end{aligned}$$

then we get a group Γ with generators $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and the following 18 (not 12 !) relators, where the twelve projective planes are printed bold:

Example 50.

$$\left\{ \begin{array}{lll} \mathbf{a_1 b_1 a_1 b_1}, & \mathbf{a_1 b_2 a_1 b_2}, & \mathbf{a_1 b_3 a_1 b_3}, \\ a_1 b_3^{-1} a_4 b_2^{-1}, & a_1 b_2^{-1} a_2 b_1^{-1}, & a_1 b_1^{-1} a_3 b_3^{-1}, \\ \mathbf{a_2 b_1 a_2 b_1}, & \mathbf{a_2 b_2 a_2 b_2}, & a_2 b_3 a_4^{-1} b_1^{-1}, \\ \mathbf{a_2 b_3^{-1} a_2 b_3^{-1}}, & a_2 b_2^{-1} a_3^{-1} b_3, & \mathbf{a_3 b_1 a_3 b_1}, \\ \mathbf{a_3 b_3 a_3 b_3}, & \mathbf{a_3 b_2^{-1} a_3 b_2^{-1}}, & a_3 b_1^{-1} a_4^{-1} b_2, \\ \mathbf{a_4 b_2 a_4 b_2}, & \mathbf{a_4 b_3 a_4 b_3}, & \mathbf{a_4 b_1^{-1} a_4 b_1^{-1}} \end{array} \right\}.$$

Theorem 50. *Let $\Gamma_0 := \ker(\Gamma \to \mathbb{Z}_2^2, a_i \mapsto (1,0), b_j \mapsto (0,1))$, generalizing the definition of the subgroup Γ_0 of a $(2m, 2n)$-group Γ.*

(1) $\Gamma^{ab} \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4$.

(2) $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2$.

(3) $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(4) *Γ has the (vertical) amalgam decomposition*

$$\Gamma \cong F_3 *_{F_{17}} (\mathbb{Z}_2^{*12} * F_3).$$

(5) *Γ_0 has the (vertical) amalgam decomposition*

$$\Gamma_0 \cong F_5 *_{F_{33}} F_5,$$

in particular Γ_0 is torsion-free and Γ is virtually torsion-free.

Proof. (1), (2), (3): GAP ([28]).

(4), (5): See Appendix B.1.

□

Remark. Taking a generalized definition of ρ_h, ρ_v, P_h, P_v, we get

$$\begin{aligned}\rho_v(b_1) &= (1,7,2,4,5,6,3,8),\\ \rho_v(b_2) &= (1,5,4,3,6,7,2,8),\\ \rho_v(b_3) &= (1,6,3,2,7,5,4,8),\end{aligned}$$

$$\begin{aligned}\rho_h(a_1) &= (1,5,2,4,3,6),\\ \rho_h(a_2) &= (1,3,4,5,2,6),\\ \rho_h(a_3) &= (1,4,3,2,5,6),\\ \rho_h(a_4) &= (1,6,4,3,5,2),\end{aligned}$$

generating $P_h \cong \mathrm{PGL}_2(7) < S_8$ and $P_v \cong \mathrm{PGL}_2(5) < S_6$ respectively.

We can take the six relators of Γ in Example 50 which are not projective planes and embed them in a $(\mathrm{PGL}_2(7), \mathrm{PGL}_2(5))$-group as follows:

Example 51.

$$R(4,3) := \left\{ \begin{array}{llll} a_1b_1a_4^{-1}b_1, & a_1b_2a_3^{-1}b_2, & a_1b_3a_2^{-1}b_3, & \underline{a_1b_3^{-1}a_4b_2^{-1}}, \\ \underline{a_1b_2^{-1}a_2b_1^{-1}}, & \underline{a_1b_1^{-1}a_3b_3^{-1}}, & a_2b_1a_3b_1, & a_2b_2a_4b_2, \\ \underline{a_2b_3a_4^{-1}b_1^{-1}}, & \underline{a_2b_2^{-1}a_3^{-1}b_3}, & a_3b_3a_4b_3, & \underline{a_3b_1^{-1}a_4^{-1}b_2} \end{array} \right\}.$$

Theorem 51. (1) *$P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.*

(2) *$P_h(X_0) \cong \mathrm{PSL}_2(7)$, $P_v(X_0) \cong \mathrm{PSL}_2(5)$, independent of the four vertices of X_0.*

(3) *$\Gamma^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3^3$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_9^3$, $\Gamma_0^{ab} \cong \mathbb{Z}_3^4$.*

Proof. (1)

$$\begin{aligned}\rho_v(b_1) &= (1,7,3,8,5,6,2,4),\\ \rho_v(b_2) &= (1,5,2,8,6,7,4,3),\\ \rho_v(b_3) &= (1,6,4,8,7,5,3,2),\end{aligned}$$

$$\begin{aligned}\rho_h(a_1) &= (1,5,2,4,3,6),\\ \rho_h(a_2) &= (1,3,4,5,2,6),\\ \rho_h(a_3) &= (1,4,3,2,5,6),\\ \rho_h(a_4) &= (1,6,4,3,5,2).\end{aligned}$$

(2) GAP ([28]).

(3) GAP ([28]).

□

5.4.3 $p \equiv 3 \pmod 8$

Let $p \equiv 3 \pmod 8$, $l \equiv 1 \pmod 4$ be primes,

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1, \operatorname{Re}(x) > 0, |x|^2 = p\},$$

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0, \operatorname{Re}(y) > 0, |y|^2 = l\}$$

and

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \rangle.$$

For example $\Gamma_{3,5}$ is given by

$$\begin{aligned} a_1 &= \psi(1 + j + k), \\ a_2 &= \psi(1 + j - k), \end{aligned}$$

$$\begin{aligned} b_1 &= \psi(1 + 2i), \\ b_2 &= \psi(1 + 2j), \\ b_3 &= \psi(1 + 2k). \end{aligned}$$

Example 52.

$$R(2,3) := \left\{ \begin{array}{ll} a_1b_1a_2b_2, & a_1b_2a_2b_1^{-1}, \\ & \\ a_1b_3a_2^{-1}b_1, & a_1b_3^{-1}a_1b_2^{-1}, \\ & \\ a_1b_1^{-1}a_2^{-1}b_3, & a_2b_3a_2b_2^{-1} \end{array} \right\}.$$

See Appendix D.8 for the GAP-program([28]) constructing $\Gamma_{3,5}$.

Theorem 52. (1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.

(3a)

$$\begin{aligned} \Gamma / \langle\langle a_1^8, (a_1b_1)^7, (b_1b_2)^3 \rangle\rangle_\Gamma &\cong \mathrm{PGL}_2(7), \\ \Gamma / \langle\langle a_1^5, a_2^5, b_1^6, (a_1b_1)^3 \rangle\rangle_\Gamma &\cong \mathrm{PSL}_2(11), \\ \Gamma / \langle\langle a_1^7, a_2^7, (a_1b_1)^4 \rangle\rangle_\Gamma &\cong \mathrm{PGL}_2(13). \end{aligned}$$

(3b)

$$\begin{aligned} \langle\langle a_1^8, (a_1b_1)^7, (b_1b_2)^3 \rangle\rangle_\Gamma^{ab} &\cong \mathbb{Z}_{14} \times \mathbb{Z}_{56}^2, \\ \langle\langle a_1^5, a_2^5, b_1^6, (a_1b_1)^3 \rangle\rangle_\Gamma^{ab} &\cong \mathbb{Z}_2 \times \mathbb{Z}_{22} \times \mathbb{Z}_{44}^2. \end{aligned}$$

(4) $U(\mathbb{H}(\mathbb{Z}[1/3, 1/5]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/5]))$ *has a presentation with generators* $a_1, a_2, b_1, b_2, b_3, i, j$ *and relators*

$$R(2,3),\ a_1ia_1i^{-1},\ a_1ja_2^{-1}j^{-1},\ b_1ib_1^{-1}i^{-1},\ b_1jb_1j^{-1},\ i^2,\ j^2,\ iji^{-1}j^{-1}.$$

(5) $(U(\mathbb{H}(\mathbb{Z}[1/3, 1/5]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/5])))^{ab} \cong \mathbb{Z}_2^4$.

(6) $\mathrm{Aut}(X) \cong D_4$.

(7) Γ *is commutative transitive.*

(8) *If* $a \in \{a_1, a_2, a_2^{-1}, a_1^{-1}\}$ *and* $b \in \{b_1, b_2, b_3, b_3^{-1}, b_2^{-1}, b_1^{-1}\}$, *then* $\langle a, b\rangle$ *is an anti-torus in* Γ.

(9) $\langle a_1, b_1\rangle \neq F_2$.

(10) $\Gamma < \mathrm{SO}_3(\mathbb{Q})$.

(11) $Z_\Gamma(a_i) = N_\Gamma(\langle a_i\rangle) = \langle a_i\rangle$, *if* $a_i \in \{a_1, a_2\}$,
$Z_\Gamma(b_j) = N_\Gamma(\langle b_j\rangle) = \langle b_j\rangle$, *if* $b_j \in \{b_1, b_2, b_3\}$.

(12) Γ *has amalgam decompositions*

$$F_3 *_{F_9} F_5 \cong \Gamma \cong F_2 *_{F_7} F_4.$$

Proof. (1)

$$\begin{aligned}\rho_v(b_1) &= (1,3,4,2),\\ \rho_v(b_2) &= (1,4,2,3),\\ \rho_v(b_3) &= (1,4,3,2),\end{aligned}$$

$$\begin{aligned}\rho_h(a_1) &= (1,2,4,6,3,5),\\ \rho_h(a_2) &= (1,4,5,6,2,3).\end{aligned}$$

(2) GAP ([28]).

(3*a*) Let q be an odd prime distinct from p and l, and choose $c, d \in \mathbb{Z}$ such that $c^2 + d^2 + 1 \equiv 0 \pmod q$, then we can define exactly as described in Proposition 48(3) a homomorphism $\tau = \tau_{c,d} : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ by

$$\tau_{c,d}(\gamma) = \left[\begin{pmatrix} x_0 + x_1 c + x_3 d + q\mathbb{Z} & -x_1 d + x_2 + x_3 c + q\mathbb{Z} \\ -x_1 d - x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c - x_3 d + q\mathbb{Z} \end{pmatrix}\right].$$

For $q = 7$ we have $\tau_{2,3} : \Gamma_{3,5} \twoheadrightarrow \mathrm{PGL}_2(7)$ given by

$$\begin{aligned} a_1 &\mapsto \left[\begin{pmatrix} 4+7\mathbb{Z} & 3+7\mathbb{Z} \\ 1+7\mathbb{Z} & 5+7\mathbb{Z}\end{pmatrix}\right],\\ a_2 &\mapsto \left[\begin{pmatrix} 5+7\mathbb{Z} & 6+7\mathbb{Z} \\ 4+7\mathbb{Z} & 4+7\mathbb{Z}\end{pmatrix}\right],\end{aligned}$$

$$\begin{aligned} b_1 &\mapsto \left[\begin{pmatrix} 5+7\mathbb{Z} & 1+7\mathbb{Z} \\ 1+7\mathbb{Z} & 4+7\mathbb{Z}\end{pmatrix}\right],\\ b_2 &\mapsto \left[\begin{pmatrix} 1+7\mathbb{Z} & 2+7\mathbb{Z} \\ 5+7\mathbb{Z} & 1+7\mathbb{Z}\end{pmatrix}\right],\\ b_3 &\mapsto \left[\begin{pmatrix} 0+7\mathbb{Z} & 4+7\mathbb{Z} \\ 4+7\mathbb{Z} & 2+7\mathbb{Z}\end{pmatrix}\right].\end{aligned}$$

In the same way $\tau_{1,3} : \Gamma_{3,5} \twoheadrightarrow \mathrm{PSL}_2(11)$ is defined by

$$\begin{aligned} a_1 &\mapsto \left[\begin{pmatrix} 4+11\mathbb{Z} & 2+11\mathbb{Z} \\ 0+11\mathbb{Z} & 9+11\mathbb{Z}\end{pmatrix}\right],\\ a_2 &\mapsto \left[\begin{pmatrix} 9+11\mathbb{Z} & 0+11\mathbb{Z} \\ 9+11\mathbb{Z} & 4+11\mathbb{Z}\end{pmatrix}\right],\end{aligned}$$

$$b_1 \mapsto \left[\begin{pmatrix} 3+11\mathbb{Z} & 5+11\mathbb{Z} \\ 5+11\mathbb{Z} & 10+11\mathbb{Z} \end{pmatrix}\right],$$
$$b_2 \mapsto \left[\begin{pmatrix} 1+11\mathbb{Z} & 2+11\mathbb{Z} \\ 9+11\mathbb{Z} & 1+11\mathbb{Z} \end{pmatrix}\right],$$
$$b_3 \mapsto \left[\begin{pmatrix} 7+11\mathbb{Z} & 2+11\mathbb{Z} \\ 2+11\mathbb{Z} & 6+11\mathbb{Z} \end{pmatrix}\right]$$

and $\tau_{0,5} : \Gamma_{3,5} \twoheadrightarrow \mathrm{PGL}_2(13)$ by

$$a_1 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 1+13\mathbb{Z} \\ 12+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right],$$
$$a_2 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 1+13\mathbb{Z} \\ 12+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right],$$

$$b_1 \mapsto \left[\begin{pmatrix} 1+13\mathbb{Z} & 3+13\mathbb{Z} \\ 3+13\mathbb{Z} & 1+13\mathbb{Z} \end{pmatrix}\right],$$
$$b_2 \mapsto \left[\begin{pmatrix} 1+13\mathbb{Z} & 2+13\mathbb{Z} \\ 11+13\mathbb{Z} & 1+13\mathbb{Z} \end{pmatrix}\right],$$
$$b_3 \mapsto \left[\begin{pmatrix} 11+13\mathbb{Z} & 0+13\mathbb{Z} \\ 0+13\mathbb{Z} & 4+13\mathbb{Z} \end{pmatrix}\right].$$

($3b$) quotpic ([59])

(4) Same idea as in Theorem 41(5) using

$$U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))/ZU(\mathbb{H}(\mathbb{Z}[1/p, 1/l])) \cong \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0\}.$$

(5) Follows from (4).

(6) GAP ([28]). $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_1, a_2^{-1}, b_1^{-1}, b_3, b_2),$$
$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_2, a_1^{-1}, b_1, b_3^{-1}, b_2).$$

(7) We can adapt Lemma 50 and Proposition 51, using Lemma 41(3). The only difference here is that possibly $\psi(x) \in \Gamma$ for some $x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\}$ such that $\mathrm{Re}(x) = 0$. But then

$$\psi(x)^2 = \psi(x_1 i + x_2 j + x_3 k)^2 = \psi(-x_1^2 - x_2^2 - x_3^2) = 1,$$

hence $\psi(x) = 1$.

(8) See Section 5.6 for the definition of an anti-torus. The statement is an application of Proposition 57 in Section 5.6 using (7) and an adaption of Lemma 50.

(9) We have $b_1 a_1^3 b_1^2 a_1 b_1^{-1} a_1^{-3} b_1^{-2} a_1^{-1} = 1$ in Γ. This also follows from $yx^3y^2xy^{-1}x^{-3}y^{-2}x^{-1} = 1$, where $x = 1 + j + k$, $y = 1 + 2i$. The statement can also be deduced from Table 24. There seems to be no smaller non-trivial freely reduced relation in $\langle x, y\rangle$ than the one of length 14 given above.

(10) A generalization of Proposition 48(2) gives an injective group homomorphism $\Gamma \to \mathrm{SO}_3(\mathbb{Q})$, defined by

$$a_1 \mapsto \frac{1}{3}\begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix},$$

$$a_2 \mapsto \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix},$$

$$b_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix},$$

$$b_2 \mapsto \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix},$$

$$b_3 \mapsto \begin{pmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(11) This follows from Proposition 8.

(12) Use [69, Theorem I.1.18]. The explicit amalgam decompositions are described in Appendix B.2.

□

See Table 23 for the orders of some quotients of Γ.

$\lvert\Gamma/\langle\langle w^k\rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6
$w=a_1$	8	64	8	512	10560	64
$w=a_2$	8	64	8	512	10560	64
$w=b_1$	16	128	16	1024	109440	168960
$w=b_2$	16	128	16	1024	109440	168960
$w=b_2$	16	128	16	1024	109440	168960

Table 23: Order of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$, $w \in \{a_1, a_2, b_1, b_2, b_3\}$, $k = 1, \ldots, 6$, in Example 52

See Table 24 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient G/U (if U is normal in G), where $U = \langle a, b\rangle$, $a \in \{a_1, a_1^2, a_2, a_2^2\}$, $b \in \{b_1, b_1^2, b_2, b_2^2, b_3, b_3^2\}$.

5.5 Some conjectures

Based on computations on the 130 examples described in the following list, we give some conjectures afterwards.

p	l	types	Sect./Ex.	P_h	$\left(\frac{l}{p}\right)$	P_v	$\left(\frac{p}{l}\right)$	Γ^{ab}	$[\Gamma,\Gamma]^{ab}$	Γ_0^{ab}

			5.2							
5	13	(o_0, o_0)	41	$\mathrm{PGL}_2(5)$	-1	$\mathrm{PGL}_2(13)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	17	(o_0, o_0)		$\mathrm{PGL}_2(5)$	-1	$\mathrm{PGL}_2(17)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	29	(o_0, o_0)		$\mathrm{PSL}_2(5)$	1	$\mathrm{PSL}_2(29)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	37	(o_0, o_0)		$\mathrm{PGL}_2(5)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	41	(o_0, o_0)		$\mathrm{PSL}_2(5)$	1	$\mathrm{PSL}_2(41)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	53	(o_0, o_0)		$\mathrm{PGL}_2(5)$	-1	$\mathrm{PGL}_2(53)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	61	(o_0, o_0)		$\mathrm{PSL}_2(5)$	1	$\mathrm{PSL}_2(61)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	73	(o_0, o_0)		$\mathrm{PGL}_2(5)$	-1	$\mathrm{PGL}_2(73)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	89	(o_0, o_0)		$\mathrm{PSL}_2(5)$	1	$\mathrm{PSL}_2(89)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	97	(o_0, o_0)		$\mathrm{PGL}_2(5)$	-1	$\mathrm{PGL}_2(97)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	17	(o_0, o_0)	40	$\mathrm{PSL}_2(13)$	1	$\mathrm{PSL}_2(17)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	29	(o_0, o_0)		$\mathrm{PSL}_2(13)$	1	$\mathrm{PSL}_2(29)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	37	(o_0, o_0)		$\mathrm{PGL}_2(13)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
13	41	(o_0, o_0)		$\mathrm{PGL}_2(13)$	-1	$\mathrm{PGL}_2(41)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	53	(o_0, o_0)		$\mathrm{PSL}_2(13)$	1	$\mathrm{PSL}_2(53)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	61	(o_0, o_0)		$\mathrm{PSL}_2(13)$	1	$\mathrm{PSL}_2(61)$	1	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
13	73	(o_0, o_0)		$\mathrm{PGL}_2(13)$	-1	$\mathrm{PGL}_2(73)$	-1	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
13	89	(o_0, o_0)		$\mathrm{PGL}_2(13)$	-1	$\mathrm{PGL}_2(89)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	97	(o_0, o_0)		$\mathrm{PGL}_2(13)$	-1	$\mathrm{PGL}_2(97)$	-1	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
17	29	(o_0, o_0)		$\mathrm{PGL}_2(17)$	-1	$\mathrm{PGL}_2(29)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
17	37	(o_0, o_0)		$\mathrm{PGL}_2(17)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
17	41	(o_0, o_0)		$\mathrm{PGL}_2(17)$	-1	$\mathrm{PGL}_2(41)$	-1	$2^3, 8^2$	$3, 16^2, 64$	$2, 3, 8^2$
17	53	(o_0, o_0)		$\mathrm{PSL}_2(17)$	1	$\mathrm{PSL}_2(53)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
17	61	(o_0, o_0)		$\mathrm{PGL}_2(17)$	-1	$\mathrm{PGL}_2(61)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	37	(o_0, o_0)		$\mathrm{PGL}_2(29)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	41	(o_0, o_0)		$\mathrm{PGL}_2(29)$	-1	$\mathrm{PGL}_2(41)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	53	(o_0, o_0)		$\mathrm{PSL}_2(29)$	1	$\mathrm{PSL}_2(53)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	61	(o_0, o_0)		$\mathrm{PGL}_2(29)$	-1	$\mathrm{PGL}_2(61)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	73	(o_0, o_0)		$\mathrm{PGL}_2(29)$	-1	$\mathrm{PGL}_2(73)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	89	(o_0, o_0)		$\mathrm{PGL}_2(29)$	-1	$\mathrm{PGL}_2(89)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	97	(o_0, o_0)		$\mathrm{PGL}_2(29)$	-1	$\mathrm{PGL}_2(97)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
37	41	(o_0, o_0)		$\mathrm{PSL}_2(37)$	1	$\mathrm{PSL}_2(41)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
37	53	(o_0, o_0)		$\mathrm{PSL}_2(37)$	1	$\mathrm{PSL}_2(53)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
37	61	(o_0, o_0)		$\mathrm{PGL}_2(37)$	-1	$\mathrm{PGL}_2(61)$	-1	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
37	73	(o_0, o_0)		$\mathrm{PSL}_2(37)$	1	$\mathrm{PSL}_2(73)$	1	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
37	89	(o_0, o_0)		$\mathrm{PGL}_2(37)$	-1	$\mathrm{PGL}_2(89)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
41	53	(o_0, o_0)		$\mathrm{PGL}_2(41)$	-1	$\mathrm{PGL}_2(53)$	-1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
41	61	(o_0, o_0)		$\mathrm{PSL}_2(41)$	1	$\mathrm{PSL}_2(61)$	1	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
73	97	(o_0, o_0)		$\mathrm{PSL}_2(73)$	1	$\mathrm{PSL}_2(97)$	1	$2^3, 3, 8^2$	?	$2, 3, 8^2$
			5.3.1							
7	23	(e_1, e_1)	42	$\mathrm{PSL}_2(7)$	1	$\mathrm{PGL}_2(23)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
7	31	(e_1, e_1)	43	$\mathrm{PGL}_2(7)$	-1	$\mathrm{PSL}_2(31)$	1	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
7	47	(e_1, e_1)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PSL}_2(47)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	31	(e_1, e_1)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PGL}_2(31)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	47	(e_1, e_1)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PGL}_2(47)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$

31	47	(e_1, e_1)		$\mathrm{PSL}_2(31)$	1	$\mathrm{PGL}_2(47)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
			5.3.2							
7	23	(e_0, e_0)	44	$\mathrm{PSL}_2(7)$	1	$\mathrm{PGL}_2(23)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
7	31	(e_0, e_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PSL}_2(31)$	1	$2^3, 3, 4$	$2^2, 4, 16^2$	$2, 3, 8^2$
7	47	(e_0, e_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PSL}_2(47)$	1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	31	(e_0, e_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PGL}_2(31)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	47	(e_0, e_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PGL}_2(47)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
31	47	(e_0, e_0)		$\mathrm{PSL}_2(31)$	1	$\mathrm{PGL}_2(47)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
			5.3.3							
3	11	(e_1, e_1)	45	$\mathrm{PGL}_2(3)$	-1	$\mathrm{PSL}_2(11)$	1	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	19	(e_1, e_1)		$\mathrm{PSL}_2(3)$	1	$\mathrm{PGL}_2(19)$	-1	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	43	(e_1, e_1)		$\mathrm{PSL}_2(3)$	1	$\mathrm{PGL}_2(43)$	-1	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	59	(e_1, e_1)		$\mathrm{PGL}_2(3)$	-1	$\mathrm{PSL}_2(59)$	1	$2, 8^2$	$8^2, 64$	$2, 8^2$
11	19	(e_1, e_1)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PSL}_2(19)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	43	(e_1, e_1)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PSL}_2(43)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	59	(e_1, e_1)		$\mathrm{PSL}_2(11)$	1	$\mathrm{PGL}_2(59)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	43	(e_1, e_1)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PGL}_2(43)$	-1	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
19	59	(e_1, e_1)		$\mathrm{PGL}_2(19)$	-1	$\mathrm{PSL}_2(59)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
			5.3.4							
3	7	(e_1, e_1)	46	$\mathrm{PSL}_2(3)$	1	$\mathrm{PGL}_2(7)$	-1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	23	(e_1, e_1)		$\mathrm{PGL}_2(3)$	-1	$\mathrm{PSL}_2(23)$	1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	31	(e_1, e_1)		$\mathrm{PSL}_2(3)$	1	$\mathrm{PGL}_2(31)$	-1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	47	(e_1, e_1)		$\mathrm{PGL}_2(3)$	-1	$\mathrm{PSL}_2(47)$	1	$2, 4^2$	$8^2, 16$	$2, 8^2$
11	7	(e_1, e_1)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PSL}_2(7)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	23	(e_1, e_1)		$\mathrm{PSL}_2(11)$	1	$\mathrm{PGL}_2(23)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	31	(e_1, e_1)		$\mathrm{PSL}_2(11)$	1	$\mathrm{PGL}_2(31)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	47	(e_1, e_1)		$\mathrm{PSL}_2(11)$	1	$\mathrm{PGL}_2(47)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	7	(e_1, e_1)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PGL}_2(7)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	23	(e_1, e_1)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PGL}_2(23)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	31	(e_1, e_1)		$\mathrm{PGL}_2(19)$	-1	$\mathrm{PSL}_2(31)$	1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	47	(e_1, e_1)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PGL}_2(47)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	7	(e_1, e_1)		$\mathrm{PGL}_2(43)$	-1	$\mathrm{PSL}_2(7)$	1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	23	(e_1, e_1)		$\mathrm{PSL}_2(43)$	1	$\mathrm{PGL}_2(23)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	31	(e_1, e_1)		$\mathrm{PSL}_2(43)$	1	$\mathrm{PGL}_2(31)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	47	(e_1, e_1)		$\mathrm{PSL}_2(43)$	1	$\mathrm{PGL}_2(47)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
			5.4.1							
7	5	(e_1, o_0)	47	$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(5)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
7	13	(e_1, o_0)	48	$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(13)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
7	17	(e_1, o_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(17)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
7	29	(e_1, o_0)		$\mathrm{PSL}_2(7)$	1	$\mathrm{PSL}_2(29)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
7	37	(e_1, o_0)		$\mathrm{PSL}_2(7)$	1	$\mathrm{PSL}_2(37)$	1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
7	41	(e_1, o_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(41)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
7	73	(e_1, o_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(73)$	-1	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
23	5	(e_1, o_0)		$\mathrm{PGL}_2(23)$	-1	$\mathrm{PGL}_2(5)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	13	(e_1, o_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PSL}_2(13)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	17	(e_1, o_0)		$\mathrm{PGL}_2(23)$	-1	$\mathrm{PGL}_2(17)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$

23	29	(e_1, o_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PSL}_2(29)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	37	(e_1, o_0)		$\mathrm{PGL}_2(23)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	41	(e_1, o_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PSL}_2(41)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	73	(e_1, o_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PSL}_2(73)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
31	5	(e_1, o_0)		$\mathrm{PSL}_2(31)$	1	$\mathrm{PSL}_2(5)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
31	13	(e_1, o_0)		$\mathrm{PGL}_2(31)$	-1	$\mathrm{PGL}_2(13)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
31	17	(e_1, o_0)		$\mathrm{PGL}_2(31)$	-1	$\mathrm{PGL}_2(17)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
31	29	(e_1, o_0)		$\mathrm{PGL}_2(31)$	-1	$\mathrm{PGL}_2(29)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
31	37	(e_1, o_0)		$\mathrm{PGL}_2(31)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
31	41	(e_1, o_0)		$\mathrm{PSL}_2(31)$	1	$\mathrm{PSL}_2(41)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
			5.4.2							
7	17	(e_0, o_0)	49	$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(17)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	17	(e_0, o_0)		$\mathrm{PGL}_2(23)$	-1	$\mathrm{PGL}_2(17)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
31	17	(e_0, o_0)		$\mathrm{PGL}_2(31)$	-1	$\mathrm{PGL}_2(17)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
7	41	(e_0, o_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(41)$	-1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	41	(e_0, o_0)		$\mathrm{PSL}_2(23)$	1	$\mathrm{PSL}_2(41)$	1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
31	41	(e_0, o_0)		$\mathrm{PSL}_2(31)$	1	$\mathrm{PSL}_2(41)$	1	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
7	73	(e_0, o_0)		$\mathrm{PGL}_2(7)$	-1	$\mathrm{PGL}_2(73)$	-1	$2^3, 3, 4$	$2^2, 4, 16^2$	$2, 3, 8^2$
			5.4.3							
3	5	(e_1, o_0)	52	$\mathrm{PGL}_2(3)$	-1	$\mathrm{PGL}_2(5)$	-1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	13	(e_1, o_0)		$\mathrm{PSL}_2(3)$	1	$\mathrm{PSL}_2(13)$	1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	17	(e_1, o_0)		$\mathrm{PGL}_2(3)$	-1	$\mathrm{PGL}_2(17)$	-1	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	29	(e_1, o_0)		$\mathrm{PGL}_2(3)$	-1	$\mathrm{PGL}_2(29)$	-1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	37	(e_1, o_0)		$\mathrm{PSL}_2(3)$	1	$\mathrm{PSL}_2(37)$	1	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	41	(e_1, o_0)		$\mathrm{PGL}_2(3)$	-1	$\mathrm{PGL}_2(41)$	-1	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	73	(e_1, o_0)		$\mathrm{PSL}_2(3)$	1	$\mathrm{PSL}_2(73)$	1	$2, 8^2$	$8^2, 64$	$2, 8^2$
11	5	(e_1, o_0)		$\mathrm{PSL}_2(11)$	1	$\mathrm{PSL}_2(5)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	13	(e_1, o_0)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PGL}_2(13)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	17	(e_1, o_0)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PGL}_2(17)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	29	(e_1, o_0)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PGL}_2(29)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	37	(e_1, o_0)		$\mathrm{PSL}_2(11)$	1	$\mathrm{PSL}_2(37)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	41	(e_1, o_0)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PGL}_2(41)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	73	(e_1, o_0)		$\mathrm{PGL}_2(11)$	-1	$\mathrm{PGL}_2(73)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	5	(e_1, o_0)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PSL}_2(5)$	1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	13	(e_1, o_0)		$\mathrm{PGL}_2(19)$	-1	$\mathrm{PGL}_2(13)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	17	(e_1, o_0)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PSL}_2(17)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	29	(e_1, o_0)		$\mathrm{PGL}_2(19)$	-1	$\mathrm{PGL}_2(29)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	37	(e_1, o_0)		$\mathrm{PGL}_2(19)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	41	(e_1, o_0)		$\mathrm{PGL}_2(19)$	-1	$\mathrm{PGL}_2(41)$	-1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	73	(e_1, o_0)		$\mathrm{PSL}_2(19)$	1	$\mathrm{PSL}_2(73)$	1	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
43	5	(e_1, o_0)		$\mathrm{PGL}_2(43)$	-1	$\mathrm{PGL}_2(5)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	13	(e_1, o_0)		$\mathrm{PSL}_2(43)$	1	$\mathrm{PSL}_2(13)$	1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	17	(e_1, o_0)		$\mathrm{PSL}_2(43)$	1	$\mathrm{PSL}_2(17)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
43	29	(e_1, o_0)		$\mathrm{PGL}_2(43)$	-1	$\mathrm{PGL}_2(29)$	-1	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	37	(e_1, o_0)		$\mathrm{PGL}_2(43)$	-1	$\mathrm{PGL}_2(37)$	-1	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	41	(e_1, o_0)		$\mathrm{PSL}_2(43)$	1	$\mathrm{PSL}_2(41)$	1	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$

	b_1	b_2	b_3	b_1^2	b_2^2	b_3^2
a_1	$4, [8,16], \mathbb{Z}_4$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$	$16, [8,64], -$	$88, [8,32], -$	$88, [8,32], -$
a_2	$4, [8,16], \mathbb{Z}_4$	$2, [8,8], \mathbb{Z}_2$	$2, [8,8], \mathbb{Z}_2$	$16, [8,64], -$	$88, [8,32], -$	$88, [8,32], -$
a_1^2	$16, [16,32], -$	$8, [16,16], -$	$8, [16,16], -$	$896, [32,64], -$	$352, [32,32], -$	$352, [32,32], -$
a_2^2	$16, [16,32], -$	$8, [16,16], -$	$8, [16,16], -$	$896, [32,64], -$	$352, [32,32], -$	$352, [32,32], -$

Table 24: $[\Gamma : U], U^{ab}, G/U$, where $U = \langle a, b\rangle$ in Example 52

Conjecture 28. *Let p, l be odd distinct primes and $\Gamma = \Gamma_{p,l}$ as in Section 5.2, 5.3.1, 5.3.3, 5.3.4, 5.4.1 or 5.4.3.*

(1) *(cf. Conjecture 24 and 25) Assume that $p, l \equiv 1 \pmod 4$ (Section 5.2).*
If $p, l \equiv 1 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_{16}^2 \times \mathbb{Z}_{64}) & \textit{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2^3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_3 \times \mathbb{Z}_{16}^2 \times \mathbb{Z}_{64}) & \textit{else} \end{cases}$$

If $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3,\ \mathbb{Z}_2^2 \times \mathbb{Z}_{16}^3) & \textit{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^3,\ \mathbb{Z}_3 \times \mathbb{Z}_{16}^3) & \textit{else} \end{cases}$$

(2) *Assume that $p, l \equiv 3 \pmod 4$ (Section 5.3.1, 5.3.3 and 5.3.4).*
If $p \pmod 8 = l \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \textit{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \textit{if } p = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \textit{else} \end{cases}$$

If $p \pmod 8 \neq l \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \textit{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \textit{if } p = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \textit{else} \end{cases}$$

(3) *Assume that $p \equiv 3 \pmod 4$ and $l \equiv 1 \pmod 4$ (Section 5.4.1 and 5.4.3).*
If $l \equiv 1 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \textit{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \textit{if } p = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \textit{else} \end{cases}$$

If $l \equiv 5 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \textit{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \textit{if } p = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \textit{else} \end{cases}$$

Conjecture 29. *Let $\Gamma = \Gamma_{p,l,e_0}$ be as in Section 5.3.2 or 5.4.2, then*

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2^3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2) & \text{else} \end{cases}$$

Conjecture 30. *Let Γ be any $\Gamma_{p,l}$ or Γ_{p,l,e_0} of Section 5, then*

$$\Gamma_0^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_8^2, & \text{if } p = 3 \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \text{else} \end{cases}$$

Remark. Note that $\Gamma_0 = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r}l^{2s}; r, s \in \mathbb{N}_0\}$ in all cases of Section 5.

Conjecture 31. *Let Γ be any $\Gamma_{p,l}$ or Γ_{p,l,e_0} of Section 5, then*

(1)

$$P_h \cong \begin{cases} \mathrm{PSL}_2(p), & \text{if } \left(\frac{l}{p}\right) = 1 \\ \mathrm{PGL}_2(p), & \text{if } \left(\frac{l}{p}\right) = -1 \end{cases}$$

and

$$P_v \cong \begin{cases} \mathrm{PSL}_2(l), & \text{if } \left(\frac{p}{l}\right) = 1 \\ \mathrm{PGL}_2(l), & \text{if } \left(\frac{p}{l}\right) = -1 \end{cases}$$

(2)

$$\left|P_h^{(k)}\right| = |P_h| \cdot p^{3(k-1)}$$

and

$$\left|P_v^{(k)}\right| = |P_v| \cdot l^{3(k-1)}.$$

As a consequence of (1) and (2):

(3)

$$\left|P_h^{(k)}\right| = \begin{cases} p^{3k-2}(p^2-1)/2, & \text{if } \left(\frac{l}{p}\right) = 1 \\ p^{3k-2}(p^2-1), & \text{if } \left(\frac{l}{p}\right) = -1 \end{cases}$$

and

$$\left|P_v^{(k)}\right| = \begin{cases} l^{3k-2}(l^2-1)/2, & \text{if } \left(\frac{p}{l}\right) = 1 \\ l^{3k-2}(l^2-1), & \text{if } \left(\frac{p}{l}\right) = -1 \end{cases}$$

Conjecture 32. *Let Γ be any $\Gamma_{p,l}$ or Γ_{p,l,e_0} of Section 5, then*

$$|K_h| = p^2$$

and

$$|K_v| = l^2.$$

Remark. In all examples of the long list above (except for $p = 73$, $l = 97$, where we were not able to compute $[\Gamma,\Gamma]^{ab}$) we have checked Conjecture 28, 29, 30, 31(1), and Conjecture 31(2) for $k = 2$.

5.6 Anti-tori

Definition. Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R(m,n) \rangle$ be a $(2m, 2n)$-group and let $a \in \langle a_1, \dots, a_m \rangle$, $b \in \langle b_1, \dots, b_n \rangle$ be two elements. The subgroup $\langle a, b \rangle < \Gamma$ is called an *anti-torus* in Γ, if a and b have no commuting non-trivial powers, i.e. if $a^r b^s \neq b^s a^r$ for all $r, s \in \mathbb{Z} \setminus \{0\}$. If moreover $\langle a, b \rangle \cong F_2$, then $\langle a, b \rangle$ is called a *free anti-torus* in Γ. Obviously, a free anti-torus is an anti-torus.

A definition in a more general context is given by Bridson and Wise in [9]:

Definition. ([9, Definition 9.1]) "Let X be a compact non-positively curved space with universal cover $p : \tilde{X} \to X$. Suppose that there is an isometrically embedded plane in $\tilde{X}$ which contains an axis for each of $\delta, \delta' \in \pi_1(X, x_0)$ and that $\tilde{x}_0 \in p^{-1}x_0$ lies in the intersection of these axes. If δ and δ' do not have powers that commute, then $\mathrm{gp}\{\delta, \delta'\}$ is called an *anti-torus*. If $\mathrm{gp}\{\delta, \delta'\}$ is free then it is called a *free anti-torus*."

Remark. The first example of a (non-free) anti-torus was given in [69] (it is $\langle a_2, b_3 \rangle$ in our Example 12). It was used to construct interesting non-residually finite groups. An existence theorem for free anti-tori (in a class not including $(2m, 2n)$-groups) appears in [9, Proposition 9.2], but no explicit example of a free anti-torus is given there or elsewhere (as far as we know).

The construction of $\Gamma_{p,l}$ in Section 5, based on the non-commutativity of quaternions, can be used to generate many anti-tori. Before giving examples, we will state some general criteria for the existence of anti-tori in commutative transitive $(2m, 2n)$-groups.

Proposition 57. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R(m,n) \rangle$ be a commutative transitive $(2m, 2n)$-group and let $a \in \langle a_1, \dots, a_m \rangle$, $b \in \langle b_1, \dots, b_n \rangle$ be two elements. Then $\langle a, b \rangle$ is an anti-torus in Γ if and only if a and b do not commute in Γ.*

Proof. Assume first that $\langle a, b \rangle$ is no anti-torus in Γ, i.e. $a^r b^s = b^s a^r$ for some $r, s \neq 0$. Obviously, a commutes with a^r, and b commutes with b^s. Using the assumption that Γ is commutative transitive, we conclude that a and b commute in Γ. The other direction follows directly from the definition of an anti-torus. □

Corollary 58. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R(m,n) \rangle$ be a commutative transitive $(2m, 2n)$-group and let $a \in \langle a_1, \dots, a_m \rangle \setminus \{1\}$, $b \in \langle b_1, \dots, b_n \rangle \setminus \{1\}$ be two non-trivial elements. Then either $\langle a, b \rangle \cong \mathbb{Z}^2$ or $\langle a, b \rangle$ is an anti-torus in Γ.*

Proof. If a and b do not commute, then $\langle a, b \rangle$ is an anti-torus in Γ by Proposition 57. Assume that a and b commute. Since Γ is torsion-free, the subgroup $\langle a, b \rangle$ is a finitely generated abelian torsion-free quotient of $\mathbb{Z}^2$. Using $a, b \neq 1$ and the uniqueness of the ab-normal forms (see Proposition 6) of powers of a and b, we conclude that $\langle a, b \rangle$ is not cyclic, but itself isomorphic to $\mathbb{Z}^2$. □

Corollary 59. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R(m,n) \rangle$ be a commutative transitive $(2m, 2n)$-group such that $(m, n) \neq (1, 1)$. Then Γ has an anti-torus.*

Proof. There are elements $a \in \{a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1}\}$ and $b \in \{b_1, \dots, b_n, b_n^{-1}, \dots, b_1^{-1}\}$ which do not commute; otherwise the $(2m, 2n)$-group Γ would be

$$\langle a_1, \dots, a_m \rangle \times \langle b_1, \dots, b_n \rangle \cong F_m \times F_n,$$

which is not commutative transitive if $(m, n) \neq (1, 1)$. By Proposition 57, $\langle a, b \rangle$ is an anti-torus in Γ. □

Reducible $(2m, 2n)$-groups have no anti-torus by the following result of Wise ([69]):

Proposition 60. *(Wise [69, Section II.4]) Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n)\rangle$ *be a* $(2m,2n)$*-group. If* Γ *has an anti-torus, then* Γ *is irreducible.*

Proof. Let $\langle a, b\rangle$ be an anti-torus, where $a \in \langle a_1, \ldots, a_m\rangle$, $b \in \langle b_1, \ldots, b_n\rangle$. Suppose that Γ is reducible. Then by [16, Proposition 1.2], the subgroup $\Lambda_1 \times \Lambda_2$ has finite index in Γ, in particular $[\langle a_1, \ldots, a_m\rangle : \Lambda_1] < \infty$ and $[\langle b_1, \ldots, b_n\rangle : \Lambda_2] < \infty$. It follows that $a^r \in \Lambda_1$, $b^s \in \Lambda_2$ for some $r, s \in \mathbb{N}$. But then $a^r b^s = b^s a^r$, a contradiction. □

Corollary 61. *Let* Γ *be a commutative transitive* $(2m, 2n)$*-group such that* $(m, n) \neq (1, 1)$*. Then* Γ *is irreducible.*

Proof. Combine Corollary 59 and Proposition 60. □

Corollary 62. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n)\rangle$ *be a commutative transitive* $(2m, 2n)$*-group and let* $b \in \langle b_1, \ldots, b_n\rangle$ *be an element such that* $Z_\Gamma(b) = \langle b\rangle$*. Then* $\langle a, b\rangle$ *is an anti-torus in* Γ *for each* $a \in \langle a_1, \ldots, a_m\rangle \setminus \{1\}$*.*

Proof. The assumption $Z_\Gamma(b) = \langle b\rangle$ implies that $b \neq 1$ and that b does not commute with any element $a \in \langle a_1, \ldots, a_m\rangle \setminus \{1\}$. Now apply Proposition 57. □

The groups $\Gamma_{p,l}$ of Section 5.2 are commutative transitive by Proposition 51, in particular we can restate the preceding results for $\Gamma_{p,l}$:

Corollary 63. *Let* $\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2)\rangle$ *be as in Section 5.2 and let* $a \in \langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle$*,* $b \in \langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle$ *be two elements. Then*

1. $\langle a, b\rangle$ *is an anti-torus in* Γ *if and only if* a *and* b *do not commute in* Γ*.*
2. *If* $a, b \neq 1$*, then either* $\langle a, b\rangle \cong \mathbb{Z}^2$ *or* $\langle a, b\rangle$ *is an anti-torus in* Γ*.*
3. Γ *has an anti-torus.*
4. Γ *is irreducible.*
5. *If* $Z_\Gamma(b) = \langle b\rangle$ *and* $a \neq 1$*, then* $\langle a, b\rangle$ *is an anti-torus in* Γ*.*

We can also restate Proposition 57 for $\Gamma_{p,l}$ only in terms of quaternions:

Proposition 64. *Let* ψ *and* $\Gamma = \Gamma_{p.l}$ *be as in Section 5.2. Assume that* $x, y \in \mathbb{H}(\mathbb{Z})$ *have type* o_0*,* $|x|^2 = p^r$*,* $|y|^2 = l^s$ *for some* $r, s \in \mathbb{N}$ *and* $xy \neq yx$*. Then* $\langle \psi(x), \psi(y)\rangle$ *is an anti-torus in* Γ*.*

Proof. By Lemma 50, $\psi(x)$ and $\psi(y)$ do not commute, hence $\langle \psi(x), \psi(y)\rangle$ is an anti-torus in Γ by Proposition 57. □

We can apply Proposition 64 for example to $\Gamma_{5,17}$ and $\Gamma_{13,17}$ or to any other $\Gamma_{p.l}$ of Section 5.2, illustrating Corollary 63(3):

Corollary 65. *Let* ψ *be as in Section 5.2.*

1. $\langle \psi(1+2i), \psi(1+4k)\rangle$ *is an anti-torus in* $\Gamma_{5,17}$*.*
2. $\langle \psi(3+2i), \psi(1+4k)\rangle$ *is an anti-torus in* $\Gamma_{13,17}$*.*
3. *Fix two distinct primes* $p, l \equiv 1 \pmod 4$*. Choose by Lemma 44(1) two quaternions* $x = x_0 + x_1 i$*,* $y = y_0 + y_3 k \in \mathbb{H}(\mathbb{Z})$ *such that* x_0, y_0 *are odd,* x_1, y_3 *are non-zero even numbers and* $|x|^2 = x_0^2 + x_1^2 = p$*,* $|y|^2 = y_0^2 + y_3^2 = l$*. Then* $\langle \psi(x), \psi(y)\rangle$ *is an anti-torus in* $\Gamma_{p,l}$*.*

Proof. (1) We apply Proposition 64, taking $p = 5$, $l = 17$, $r = 1$, $s = 1$.

(2) We apply Proposition 64, taking $p = 13$, $l = 17$, $r = 1$, $s = 1$.

(3) Apply Proposition 64, taking $r = 1$, $s = 1$ and using the fact that $x_0 + x_1 i$ and $y_0 + y_3 k$ do not commute.

$\square$

Proposition 66. *There are distinct primes $p, l \equiv 1 \pmod 4$, a group*

$$\Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2) \rangle$$

as in Section 5.2, and an element $b \in \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle$, such that $\langle a, b \rangle$ is an anti-torus for all $a \in \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle \setminus \{1\}$.

First Proof. We choose $p = 5$, $l = 13$ and $b = \psi(1 + 2i + 2j + 2k) \in \Gamma_{5,13}$. By Theorem 41(8), we have $Z_\Gamma(b) = \langle b \rangle$ and apply now Corollary 62. $\square$

Second Proof. We take $p = 5$, $l = 29$, $b = \psi(3 + 2j + 4k) \in \Gamma_{5,29}$ and $c = j + 2k$. Assume that there is an element $a \in \langle a_1, a_2, a_3 \rangle \setminus \{1\} \subset \Gamma_{5,29}$ commuting with some power b^t, $t \in \mathbb{N}$. Note that $b^t = \psi((3 + 2j + 4k)^t) = \psi(x_0 + \lambda j + 2\lambda k)$ for some x_0, $\lambda \neq 0$, depending on t. Then, applying Proposition 53 to $z = (3+2j+4k)^t$, there are $x, y \in \mathbb{Z}$ such that $\gcd(x, y) = \gcd(x, pl) = \gcd(y, pl) = 1$ and $x^2 + 4 \cdot 5y^2 = 5^r 29^s$ for some $r, s \in \mathbb{N}$. But this implies $x^2 = 5(5^{r-1}29^s - 4y^2)$, contradicting $\gcd(x, 5 \cdot 29) = 1$. (What we use here is that such a decomposition $x^2 + 4 \cdot |c|^2 y^2 = p^r l^s$ implies $\gcd(|c|^2, pl) = 1$, as already noted in [53].) $\square$

Proposition 67. *There are distinct primes $p, l \equiv 1 \pmod 4$, a group*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2) \rangle$$

as in Section 5.2, and elements $a \in \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle$, $b \in \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle \setminus \{1\}$ such that $\langle a, b_j \rangle$ is an anti-torus for all $b_j \in \{b_1, \dots, b_{\frac{l+1}{2}}\}$, but $\langle a, b \rangle$ is no anti-torus, in particular $Z_\Gamma(a) \neq \langle a \rangle$.

Proof. We take $p = 29$, $l = 41$, $a = \psi(3+4i+2j)$ and $b = \psi(-31+24i+12j) = \psi(1+6j-2k)\psi(1+6j+2k)$, hence $ab = ba$. It is easy to check that a does not commute with any $b_j \in \{b_1, \dots, b_{21}\}$, in particular $\langle a, b_j \rangle$ is an anti-torus by Proposition 57. $\square$

Also note the following corollary of Proposition 49, see Corollary 75 for a generalization to $(2m, 2n)$-groups:

Corollary 68. *Let $p, l \equiv 1 \pmod 4$ be distinct primes and*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2) \rangle.$$

Then there are always elements $a \in \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle \setminus \{1\}$ and $b \in \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle \setminus \{1\}$ such that $\langle a, b \rangle$ is no anti-torus.

The next proposition gives sufficient conditions to generate *free* anti-tori in $\Gamma_{p,l}$:

Proposition 69. *Let $p, l \equiv 1 \pmod 4$, ψ as in Section 5.2, $r, s \in \mathbb{N}$, and $x, y \in \mathbb{H}(\mathbb{Z})$ of type o_0, such that $|x|^2 = p^r$, $|y|^2 = l^s$. Suppose that x, y generate a (rank 2) free group in the multiplicative group $U(\mathbb{H}(\mathbb{Q})) = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$ (or equivalently in the subgroup $U(\mathbb{H}(\mathbb{Z}[1/p, 1/l])) < U(\mathbb{H}(\mathbb{Q}))$). Then $\langle \psi(x), \psi(y) \rangle$ is a free anti-torus in $\Gamma_{p,l}$.*

Proof. Extending ψ from the integer to the rational quaternions, let

$$\tilde{\psi} : U(\mathbb{H}(\mathbb{Q})) \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$
$$x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

where $x_0, x_1, x_2, x_3 \in \mathbb{Q}$, $x \neq 0$. Note that $U(\mathbb{H}(\mathbb{Q})) = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$ equipped with quaternion multiplication is a non-abelian group, $\tilde{\psi}$ is a group homomorphism such that

$$\ker(\tilde{\psi}) = ZU(\mathbb{H}(\mathbb{Q})) = \{x \in \mathbb{H}(\mathbb{Q}) \setminus \{0\} : x = \mathrm{Re}(x)\},$$

and $\tilde{\psi}(x) = \psi(x)$, if $x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\}$. Now, fix some x and y satisfying the assumptions made in the proposition. We restrict $\tilde{\psi}$ to the free subgroup $\langle x, y \rangle < U(\mathbb{H}(\mathbb{Q}))$:

$$\tilde{\psi}|_{\langle x,y \rangle} : \langle x, y \rangle \cong F_2 \twoheadrightarrow \langle \tilde{\psi}(x), \tilde{\psi}(y) \rangle = \langle \psi(x), \psi(y) \rangle < \Gamma_{p,l}.$$

We have

$$\ker\left(\tilde{\psi}|_{\langle x,y \rangle}\right) = \langle x, y \rangle \cap ZU(\mathbb{H}(\mathbb{Q})) < Z(\langle x, y \rangle) \cong ZF_2 = \{1\},$$

in particular $\tilde{\psi}|_{\langle x,y \rangle}$ is an isomorphism, i.e. $\langle \psi(x), \psi(y) \rangle \cong F_2$.

By construction

$$\psi(x) \in \langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle = \{\psi(x) \mid x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r,\ r \in \mathbb{N}_0\} < \Gamma_{p,l},$$

$$\psi(y) \in \langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle = \{\psi(y) \mid y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l^s,\ s \in \mathbb{N}_0\} < \Gamma_{p,l},$$

where the $(p+1, l+1)$-group $\Gamma_{p,l}$ is generated by $a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}$. This shows that $\langle \psi(x), \psi(y) \rangle$ is a free anti-torus in $\Gamma_{p,l}$. ◻

For example, if $\langle 3 + 2i, 1 + 4k \rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$, then Proposition 69 would give an explicit free anti-torus $\langle \psi(3 + 2i), \psi(1 + 4k) \rangle$ in $\Gamma_{13,17}$.

Question 12. $\langle 3 + 2i, 1 + 4k \rangle \cong F_2$*?*

More generally:

Question 13. *Let p, l be distinct odd primes. Is there a pair $x, y \in \mathbb{H}(\mathbb{Z})$ such that $\langle x, y \rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$, where $|x|^2 = p^r$, $|y|^2 = l^s$ for some $r, s \in \mathbb{N}$?*

The anti-tori constructed in Corollary 65(1) and Theorem 52(8) are not free:

Proposition 70. (1) *Let ψ be as in Section 5.2, $x = 1 + 2i$, $y = 1 + 4k$, $a = \psi(x)$, $b = \psi(y)$. Then the anti-torus $\langle a, b \rangle$ in $\Gamma_{5,17}$ is not free.*

(2) *Let ψ be as in Section 5.4.3, $x = 1 + j + k$, $y = 1 + 2i$, $a = \psi(x)$, $b = \psi(y)$. Then the anti-torus $\langle a, b \rangle$ in $\Gamma_{3,5}$ is not free.*

Proof.

(1) In $\Gamma_{5,17}$, we have the relation

$$a^3 b^2 a b^{-1} a^2 b^{-1} a^2 b^{-1} a^{-4} b^{-2} a^{-1} b a^{-2} b^{-1} a^{-8} b^{-1} a b^2 a b^{-1} a^{-2} b a^{-1} b^{-2} a^{-2} b^{-2} a^3$$
$$b a^{-2} b^2 a^2 b^2 a b^{-1} a^2 b a^{-1} b^{-2} a^{-1} b a^8 b a^2 b^{-1} a b^2 a^4 b a^{-2} b a^{-2} b a^{-1} b^{-2} a^{-5} b^{-1} a = 1.$$

To get this long relation, we have used the GAP-command ([28]) PresentationSubgroupMtc(G,U), where G and U describe Γ and its subgroup $\langle a, b\rangle$, respectively. The corresponding relation in $U(\mathbb{H}(\mathbb{Q}))$ is

$$x^3y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}x^{-8}y^{-1}xy^2xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^3$$
$$yx^{-2}y^2x^2y^2xy^{-1}x^2yx^{-1}y^{-2}x^{-1}yx^8yx^2y^{-1}xy^2x^4yx^{-2}yx^{-2}yx^{-1}y^{-2}x^{-5}y^{-1}x = 1,$$

in particular $\langle x, y\rangle \neq F_2$. Note that GAP ([28]) also shows that $[\Gamma_{5,17} : \langle a, b\rangle] = 32$ and $\langle a, b\rangle^{ab} \cong \mathbb{Z}_{16} \times \mathbb{Z}_{64}$. Moreover, $\langle a, b\rangle \cong \langle x, y\rangle/Z\langle x, y\rangle$, where $Z\langle x, y\rangle \neq 1$, since e.g.

$$xy^{-1}xy^2x^8yx^{-3}y^{-1}xyx^4y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}x^{-8}y^{-1}xy^2x$$
$$y^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^2y^{-1}x^2y^2xy^{-1}x^2yx^{-1}y^{-2}x^{-1}yx^8yx^2y^{-1}xy^2x^4yx^{-2}yx^{-2}$$
$$yx^{-1}y^{-2}x^{-4}y^{-1}x^{-1}y^{-1}x^3y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}x^{-8}y^{-1}xy^2x$$
$$y^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^5y^2xy^{-1}x^2y^{-1}x^4y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}$$
$$y^{-1}x^{-8}y^{-1}xy^2xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^2y^{-1} = \frac{1}{17^8} \in Z\langle x, y\rangle.$$

(2) See Theorem 52(9). Recall that $\langle a^t, b^t\rangle$, $t \in \mathbb{N}$, is never abelian, and $[\Gamma_{3,5} : \langle a, b\rangle] = 4$. Also note that $[\Gamma_{3,5} : \langle a^2, b^2\rangle] = 896 < \infty$, using GAP ([28]). In particular $\langle a^2, b^2\rangle$ is not free by the following remark. $\square$

Remark. If $\langle a, b\rangle$ is a free subgroup in a $(2m, 2n)$-group Γ, then the index $[\Gamma : \langle a, b\rangle]$ is infinite. Otherwise, Γ would be virtually free, but this is impossible since being virtually free is a quasi-isometry invariant (see e.g. [31, IV.50]), and using the facts that $(2m, 2n)$-groups are all quasi-isometric (to $F_2 \times F_2$), if $m, n \geq 2$ (see Proposition 80(4)), and that there are $(2m, 2n)$-groups which obviously are not virtually free, e.g. the virtually simple groups constructed in Section 3.

Note the following general question of Daniel Wise appearing in Mladen Bestvina's problem list "Questions in Geometric Group Theory" ([5]):

Question 14. *(Wise [5, Question 2.7]) "Let G act properly discontinuously and cocompactly on a CAT(0) space (or let G be automatic). Consider two elements a, b of G. Does there exist $n > 0$ such that either the subgroup $\langle a^n, b^n\rangle$ is free or $\langle a^n, b^n\rangle$ is abelian?"*

Remark. Free subgroups of $U(\mathbb{H}(\mathbb{Q}))$ also induce free subgroups in $\mathrm{SO}_3(\mathbb{Q}) < \mathrm{SO}_3(\mathbb{R})$ via the group homomorphism (see Section 5.2)

$$\begin{aligned} \vartheta : U(\mathbb{H}(\mathbb{Q})) &\to \mathrm{SO}_3(\mathbb{Q}) \\ x_0 + x_1 i + x_2 j + x_3 k &\mapsto \frac{1}{|x|^2} \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}. \end{aligned}$$

The proof is similar to a part of the proof of Proposition 69: First remember that

$$\ker(\vartheta) = ZU(\mathbb{H}(\mathbb{Q})) = \{x \in \mathbb{H}(\mathbb{Q}) \setminus \{0\} : x = \mathrm{Re}(x)\}.$$

Assume now that $F_2 \cong \langle x, y\rangle < U(\mathbb{H}(\mathbb{Q}))$. Then

$$\vartheta|_{\langle x,y\rangle} : \langle x, y\rangle \twoheadrightarrow \langle \vartheta(x), \vartheta(y)\rangle < \mathrm{SO}_3(\mathbb{Q})$$

is bijective, since it is surjective and

$$\ker\left(\vartheta|_{\langle x,y\rangle}\right) = \langle x, y\rangle \cap ZU(\mathbb{H}(\mathbb{Q})) < Z(\langle x, y\rangle) \cong ZF_2 = \{1\},$$

in particular $\langle \vartheta(x), \vartheta(y)\rangle \cong F_2$.

As an example, if $\langle 3+2i, 1+4k\rangle \cong F_2$ (see Question 12), then the two rotations (around perpendicular axes)

$$\vartheta(3+2i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/13 & -12/13 \\ 0 & 12/13 & 5/13 \end{pmatrix}, \ \vartheta(1+4k) = \begin{pmatrix} -15/17 & -\ 8/17 & 0 \\ 8/17 & -15/17 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

would generate a free subgroup in $\mathrm{SO}_3(\mathbb{Q})$. Note that if

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R((p+1)/2, (l+1)/2)\rangle$$

as in Section 5.2, then $\langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle$ and $\langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle$ are free subgroups of $\mathrm{SO}_3(\mathbb{Q})$ (combine Corollary 7(1) and Proposition 48(2), cf. [43, Corollary 2.1.11]). For example, taking $p = 5$,

$$F_3 \cong \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix}, \begin{pmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle < \mathrm{SO}_3(\mathbb{Q}).$$

However, by Proposition 70(1) and (2) respectively, the following subgroups of $\mathrm{SO}_3(\mathbb{Q})$ are not free:

$$\langle \vartheta(1+2i), \vartheta(1+4k)\rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} -15/17 & -\ 8/17 & 0 \\ 8/17 & -15/17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

$$\langle \vartheta(1+j+k), \vartheta(1+2i)\rangle = \left\langle \frac{1}{3}\begin{pmatrix} -3 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix} \right\rangle.$$

We can use the amalgam decompositions of $\Gamma_{p,l}$ to construct integer quaternions x, y generating a non-abelian free group in $U(\mathbb{H}(\mathbb{Q}))$ such that $|x|^2$ and $|y|^2$ are not both powers of the same prime number (cf. Question 13). We illustrate this with an example:

Proposition 71. *Let ψ be as in Section 5.4.3, $x = 1+2i+2j+4k$ of norm $|x|^2 = 5^2$, $y = 3-2i+j-k$ of norm $|y|^2 = 3 \cdot 5$. Then $\langle x, y\rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$.*

Proof. We have $\psi(x) = \psi(1+2i)\psi(1+2j) = b_1 b_2$ and $\psi(y) = \psi(1+j+k)\psi(1-2k) = a_1 b_3^{-1}$ in $\Gamma_{3,5}$. By the vertical amalgam decomposition of $\Gamma_{3,5}$ given in Appendix B.2

$$F_2 \cong \langle s_1, s_4\rangle = \langle b_1 b_2, a_1 b_3^{-1}\rangle = \langle \psi(x), \psi(y)\rangle < \Gamma_{3,5},$$

hence $\langle x, y\rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$. $\square$

5.7 A different quaternion construction: $p = 2$, $l = 5$

Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$. Motivated by the identities ([23])

$$\begin{aligned}
(1+i)(x_0 + x_1 i + x_2 j + x_3 k) &= (x_0 + x_1 i - x_3 j + x_2 k)(1+i), \\
(1+j)(x_0 + x_1 i + x_2 j + x_3 k) &= (x_0 + x_3 i + x_2 j - x_1 k)(1+j), \\
(1+k)(x_0 + x_1 i + x_2 j + x_3 k) &= (x_0 - x_2 i + x_1 j + x_3 k)(1+k),
\end{aligned}$$

we identify

$$\begin{aligned}
a_1 &\cong 1+i, \\
a_2 &\cong 1+j, \\
a_3 &\cong 1+k, \\
a_3^{-1} &\cong 1-k, \\
a_2^{-1} &\cong 1-j, \\
a_1^{-1} &\cong 1-i, \\
b_1 &\cong 1+2i, \\
b_2 &\cong 1+2j, \\
b_3 &\cong 1+2k, \\
b_3^{-1} &\cong 1-2k, \\
b_2^{-1} &\cong 1-2j, \\
b_1^{-1} &\cong 1-2i,
\end{aligned}$$

and get the following $(6,6)$-group:

Example 53. $\Gamma = \langle a_1, a_2, a_3, b_1, b_2, b_3 \mid R(3,3) \rangle$, *where*

$$R(3,3) := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, & a_1 b_3 a_1^{-1} b_2, \\ \\ a_2 b_1 a_2^{-1} b_3, & a_2 b_2 a_2^{-1} b_2^{-1}, & a_2 b_3 a_2^{-1} b_1^{-1}, \\ \\ a_3 b_1 a_3^{-1} b_2^{-1}, & a_3 b_2 a_3^{-1} b_1, & a_3 b_3 a_3^{-1} b_3^{-1} \end{array} \right\}.$$

Note that there is no map ψ involved in this construction, in particular Γ behaves completely differently than the groups $\Gamma_{p,l}$ constructed before, e.g. Γ is reducible, $(1+i)^4 = -4$, but $a_1^4 \neq 1_\Gamma$, $1+i$ and $1+2j$ do not commute, but $\langle a_1, b_2 \rangle$ is no anti-torus.

Theorem 53. (1) $P_h = 1$, $P_v \cong S_4 < S_6$.

(2) Γ *is reducible.*

(3) $\Lambda_1 \times \Lambda_2 \cong F_{49} \times F_3$ *and* $[\Gamma : \Lambda_1 \times \Lambda_2] = 24$.

Proof. (1)

$$\begin{aligned}
\rho_v(b_1) &= (), \\
\rho_v(b_2) &= (), \\
\rho_v(b_3) &= (),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (2,4,5,3), \\
\rho_h(a_2) &= (1,3,6,4), \\
\rho_h(a_3) &= (1,5,6,2).
\end{aligned}$$

(2) This follows from the subsequent Lemma 72(1).

(3) Apply Lemma 72(3).

□

Lemma 72. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n) \rangle$ *be a* $(2m, 2n)$*-group such that* $P_h = 1$*. Then*

(1) Γ *is reducible.*

(2) $\Lambda_1 \cong \ker \rho_h$ *and* $\Lambda_2 \cong \ker \rho_v = \langle b_1, \ldots, b_n \rangle$*.*

(3) $\Lambda_1 \times \Lambda_2 \cong F_{(m-1)|P_v|+1} \times F_n$ *has index* $|P_v|$ *in* Γ*.*

Proof. (1) By Proposition 1(2a) it is enough to show that $P_h^{(2)} = 1$. Let $b \in E_v$, $a = \hat{a} \cdot \tilde{a} \in E_h^{(2)}$, where $\hat{a}, \tilde{a} \in E_h$, $\hat{a} \neq \tilde{a}^{-1}$. Then $\rho_v(b)(\hat{a}) = \hat{a}$ and $\rho_v(\rho_h(\hat{a})(b))(\tilde{a}) = \tilde{a}$, i.e. $\rho_v^{(2)}(b)(a) = a$. See Figure 10 for an illustration.

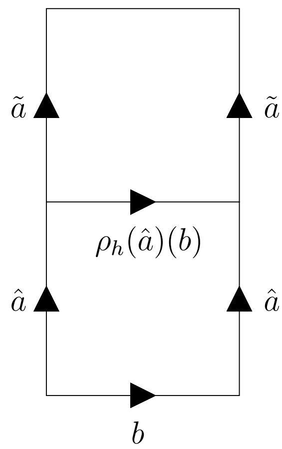


Figure 10: $P_h^{(2)} = 1$

(2) $\Lambda_1 < \ker \rho_h$ always holds, $\ker \rho_h < \Lambda_1$ follows from Lemma 26(1a) using $P_h = 1$.

(3) This follows from $[\langle a_1, \ldots, a_m \rangle : \Lambda_1] = |P_v|$, which is a direct consequence of (2).

□

6 Periodic tilings and $\mathbb{Z}^2$-subgroups

Let X be a locally compact complete CAT(0)-space and Γ a properly discontinuous and cocompact group of isometries. Then, in this general context, it is an open question if certain free abelian subgroups of Γ exist. We quote from [1, Question 2.3]:

"Is hyperbolicity equivalent to the non-existence of a subgroup of Γ isomorphic to $\mathbb{Z}^2$? More generally, does Γ contain a subgroup isomorphic to $\mathbb{Z}^k$ if X contains a k-flat?
By the work of Bangert and Schroeder [...] the answer is positive in the case of compact, real analytic Riemannian manifolds. Except for this, the answers to these questions are completely open, even in the case where X is a geodesically complete and piecewise Euclidean complex of dimension two!"

We will give in Proposition 74(3) an elementary proof that $(2m, 2n)$-groups always contain a $\mathbb{Z}^2$-subgroup. The idea of this proof (and the fact that this result holds) was explained to me by Guyan Robertson.

Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n)\rangle$ be a $(2m, 2n)$-group. Represent each of the mn geometric squares in $R(m,n)$ by an expression of the form $aba'b'$, where $a, a' \in E_h$, $b, b' \in E_v$, and let $T(\Gamma)$ be the set consisting of the $4mn$ (non-geometric) squares

$$T(\Gamma) := \bigcup_{aba'b' \in R(m,n)} \{aba'b',\ a'b'ab,\ a^{-1}b'^{-1}a'^{-1}b^{-1},\ a'^{-1}b^{-1}a^{-1}b'^{-1}\}.$$

Clearly, the definition of $T(\Gamma)$ only depends on Γ, but not on the choice of representatives in $R(m,n)$. Note that the four expressions $aba'b'$, $a'b'ab$, $a^{-1}b'^{-1}a'^{-1}b^{-1}$ and $a'^{-1}b^{-1}a^{-1}b'^{-1}$ represent the same geometric square. We always visualize them in the plane as in Figure 11. Moreover,

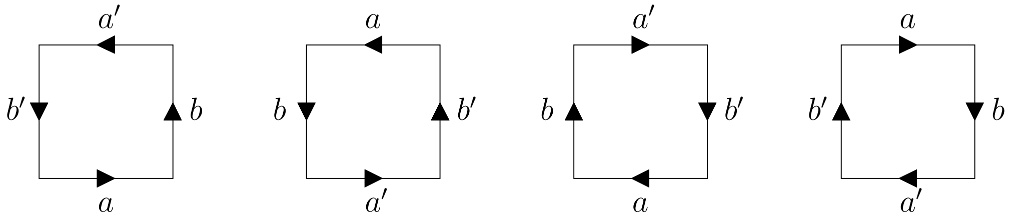


Figure 11: Tiles in $T(\Gamma)$ induced by the geometric square $aba'b'$

we assume that the four edges of each square in $T(\Gamma)$ have length 1. Such unit squares are usually called *Wang tiles* (named after Hao Wang [67]). We define "south-", "east-", "north-" and "west-functions" $S, E, N, W : T(\Gamma) \to E_h \sqcup E_v$ as follows:

$$S(aba'b') := a,\ E(aba'b') := b,\ N(aba'b') := {a'}^{-1},\ W(aba'b') := {b'}^{-1}.$$

A *tiling* (of the plane) is a map $f : \mathbb{Z}^2 \to T(\Gamma)$. We are only interested in *valid* tilings, i.e. tilings where all edges match. Precisely, this means that for each $(x,y) \in \mathbb{Z}^2$

$$S(f(x,y)) = N(f(x,y-1)),$$
$$W(f(x,y)) = E(f(x-1,y)).$$

A valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ is said to satisfy the adjacency condition (AC) if for each $(x,y) \in \mathbb{Z}^2$

$$S(f(x,y)) \neq N(f(x-1,y-1))^{-1}, \qquad \text{(AC)}$$
$$W(f(x,y)) \neq E(f(x-1,y-1))^{-1}.$$

Note that (AC) is equivalent to

$$\begin{aligned}
S(f(x-1,y))^{-1} \neq S(f(x,y)) \neq S(f(x+1,y))^{-1},\\
N(f(x-1,y))^{-1} \neq N(f(x,y)) \neq N(f(x+1,y))^{-1},\\
E(f(x,y-1))^{-1} \neq E(f(x,y)) \neq E(f(x,y+1))^{-1},\\
W(f(x,y-1))^{-1} \neq W(f(x,y)) \neq W(f(x,y+1))^{-1}.
\end{aligned}$$

for each $(x,y) \in \mathbb{Z}^2$ and it requires that any word consisting of consecutive horizontal or consecutive vertical edges in the tiling f is freely reduced, where the words of edges are seen as elements in $\langle a_1,\ldots,a_m\rangle < \Gamma$ or $\langle b_1,\ldots,b_n\rangle < \Gamma$ respectively.

We say that a valid tiling $f:\mathbb{Z}^2 \to T(\Gamma)$ satisfies the condition (AC_j) for some fixed $j \in \mathbb{Z}$, if for each $i \in \mathbb{Z}$

$$\begin{aligned}
S(f(i,i+j)) \neq N(f(i-1,i-1+j))^{-1},\\
W(f(i,i+j)) \neq E(f(i-1,i-1+j))^{-1}.
\end{aligned} \tag{AC_j}$$

Note that if (AC_j) holds in a valid tiling $f:\mathbb{Z}^2 \to T(\Gamma)$ for each $j \in \mathbb{Z}$, then also (AC) holds for f.

A valid tiling $f:\mathbb{Z}^2 \to T(\Gamma)$ is called *periodic* with period $(\tilde{a},\tilde{b}) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, if $f(x,y) = f(x+\tilde{a}, y+\tilde{b})$ for each $(x,y) \in \mathbb{Z}^2$.

The following lemma guarantees the unique extension of any $T(\Gamma)$-valued map f on the main diagonal in $\mathbb{Z}^2$ to a valid tiling of the whole plane satisfying (AC), provided f satisfies the inequalities of condition (AC_0).

Lemma 73. *Let $f:\{(i,i): i \in \mathbb{Z}\} \to T(\Gamma)$ be a map such that for each $i \in \mathbb{Z}$*

$$\begin{aligned}
S(f(i,i)) \neq N(f(i-1,i-1))^{-1},\\
W(f(i,i)) \neq E(f(i-1,i-1))^{-1}.
\end{aligned}$$

Then f uniquely extends to a valid tiling $\mathbb{Z}^2 \to T(\Gamma)$. Moreover, this tiling satisfies (AC).

Proof. The existence and uniqueness of a valid tiling $\mathbb{Z}^2 \to T(\Gamma)$ extending f follows directly by the link condition in the $(2m,2n)$-group Γ. We call this extension again f. By assumption, this f satisfies (AC_0). Let $n \in \mathbb{N}_0$, we prove now that condition (AC_n) implies condition (AC_{n+1}). In the same way, one can prove that (AC_{-n}) implies (AC_{-n-1}). By induction, we conclude that $f:\mathbb{Z}^2 \to T(\Gamma)$ satisfies (AC).

Assume now that (AC_n) holds. Fix any $i \in \mathbb{Z}$, to show (AC_{n+1}), we have to prove that

$$\begin{aligned}
S(f(i,i+n+1)) \neq N(f(i-1,i+n))^{-1},\\
W(f(i,i+n+1)) \neq E(f(i-1,i+n))^{-1}.
\end{aligned}$$

Assume first that

$$N(f(i-1,i+n))^{-1} = S(f(i,i+n+1)) \quad (= N(f(i,i+n))).$$

Since $W(f(i,i+n)) = E(f(i-1,i+n))$, it follows by the link condition in Γ that

$$S(f(i,i+n)) = S(f(i-1,i+n))^{-1} = N(f(i-1,i+n-1))^{-1},$$

contradicting (AC_n). Similarly, assume that

$$W(f(i,i+n+1)) = E(f(i-1,i+n))^{-1} = W(f(i,i+n))^{-1}.$$

Then $S(f(i,i+n+1)) = N(f(i,i+n))$ implies

$$E(f(i,i+n)) = E(f(i,i+n+1))^{-1} = W(f(i+1,i+n+1))^{-1},$$

again contradicting (AC_n). $\square$

Proposition 74. *Fix a $(2m,2n)$-group $\Gamma = \langle a_1,\dots,a_m,b_1,\dots,b_n \mid R(m,n)\rangle$ and the corresponding tile set $T(\Gamma)$ defined as above.*

(1) *There is a periodic valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying (AC).*

(2) *There is a valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying (AC), and a number $\tilde{a} \in \mathbb{N}$ such that $f(x,y) = f(x+\tilde{a},y) = f(x,y+\tilde{a})$, i.e. f has the two periods $(\tilde{a},0)$ and $(0,\tilde{a})$ and therefore is* doubly periodic.

(3) *There are commuting elements $a \in \langle a_1,\dots,a_m\rangle < \Gamma$, $b \in \langle b_1,\dots,b_n\rangle < \Gamma$ such that*

$$0 < |a| = |b| \le 64m^2n^2,$$

in particular $\langle a,b\rangle$ is a subgroup of Γ isomorphic to $\mathbb{Z}^2$.

Proof. (1) Our goal is to construct a valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$, such that $f(x,y) = f(x+2,y+2)$. Fix any square $t := aba'b' \in T(\Gamma)$ and define f periodic along the diagonal $\{(i,i) : i \in \mathbb{Z}\}$ as follows. If $a \neq a'$ and $b \neq b'$, then we define $f(i,i) = t$, $i \in \mathbb{Z}$. If $a = a'$, then we define $f(2i,2i) = t$, $f(2i+1,2i+1) = a^{-1}b'^{-1}a^{-1}b^{-1}$, $i \in \mathbb{Z}$. If $b = b'$, then we define $f(2i,2i) = t$, $f(2i+1,2i+1) = a'^{-1}b^{-1}a^{-1}b^{-1}$, $i \in \mathbb{Z}$. See Figure 12 for an illustration of these three cases. Now we can apply Lemma 73 to the map $f : \{(i,i) : i \in \mathbb{Z}\} \to T(\Gamma)$. The obtained

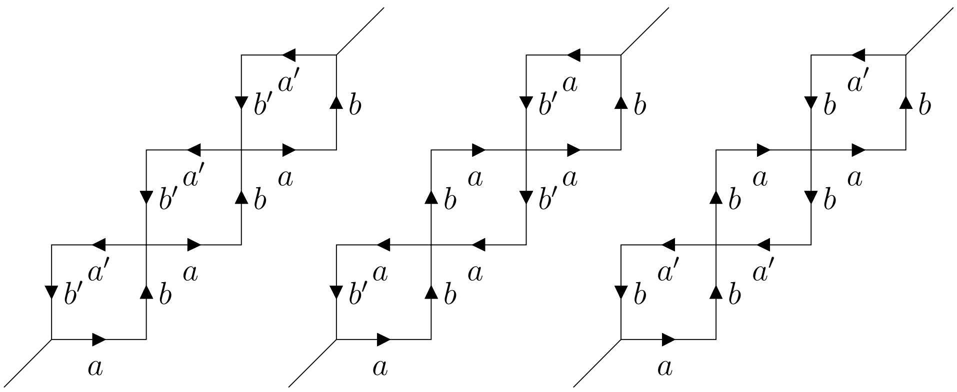


Figure 12: Definition of $f(i,i)$ in Proposition 74

unique extension $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfies (AC) and is obviously periodic with period $(2,2)$ (in the first case where $a \neq a'$ and $b \neq b'$, there is a smaller period $(1,1)$).

(2) We use an idea probably going back to Raphael M. Robinson ([61]). It was explained to me by Guyan Robertson. Let $f : \mathbb{Z}^2 \to T(\Gamma)$ be the tiling with period $(2,2)$ satisfying (AC) obtained in part (1). Since $|T(\Gamma)| = 4mn$ is finite, we have

$$|\{(f(i,-i), f(i+1,-i+1)) : i \in \mathbb{Z}\}| \le |T(\Gamma) \times T(\Gamma)| = (4mn)^2 < \infty,$$

in particular there are $i \neq j$, such that $|j-i| \leq (4mn)^2$ and

$$f(i,-i) = f(j,-j) \text{ and } f(i+1,-i+1) = f(j+1,-j+1)$$

It follows that

$$f(x,y) = f(x+j-i, y+i-j)$$

for each $(x,y) \in \mathbb{Z}^2$. Now, we compute

$$f(x,y) = f(x+j-i, y+i-j) = f(x+2j-2i, y+2i-2j) = f(x, y+4i-4j) = f(x, y+4j-4i)$$

and

$$f(x,y) = f(x+j-i, y+i-j) = f(x+2j-2i, y+2i-2j) = f(x+4j-4i, y) = f(x+4i-4j, y).$$

Note that $0 < |4j-4i| \leq 4(4mn)^2 = 64m^2n^2$.

(3) We use the doubly periodic valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying (AC) of part (2):

$$f(x,y) = f(x+\tilde{a}, y) = f(x, y+\tilde{a}),$$

where $\tilde{a} > 0$. Since the tiling only describes relations in Γ, we obviously have two commuting elements $a \in \langle a_1, \ldots, a_m \rangle$, $b \in \langle b_1, \ldots, b_n \rangle$ corresponding to the period $\tilde{a}$. Because of condition (AC), a and b are freely reduced and we therefore have $|a| = |b| = \tilde{a} \in \mathbb{N}$. The upper bound $64m^2n^2$ for the length of $|a|$ and $|b|$ can be obtained by the explicit construction in (2). The statement $\langle a, b \rangle \cong \mathbb{Z}^2$ follows as in Corollary 58.

□

Remark. $T(\Gamma)$ is a *reflection-closed* 4*-way deterministic* tile set (following the terminology of [37]), but $T(\Gamma)$ is never *aperiodic* by Proposition 74(1).

We want to illustrate the constructions in the proof of Proposition 74 with a concrete example and take Example 52 given as

$$\Gamma = \Gamma_{3,5} = \langle a_1, a_2, b_1, b_2, b_3 \mid a_1b_1a_2b_2, a_1b_2a_2b_1^{-1}, a_1b_3a_2^{-1}b_1, a_1b_3^{-1}a_1b_2^{-1}, a_1b_1^{-1}a_2^{-1}b_3, a_2b_3a_2b_2^{-1} \rangle.$$

This defines the tile set

$$\begin{aligned}
T(\Gamma) = &\{a_1b_1a_2b_2,\ a_2b_2a_1b_1,\ a_1^{-1}b_2^{-1}a_2^{-1}b_1^{-1},\ a_2^{-1}b_1^{-1}a_1^{-1}b_2^{-1}\} \\
\cup &\{a_1b_2a_2b_1^{-1},\ a_2b_1^{-1}a_1b_2,\ a_1^{-1}b_1a_2^{-1}b_2^{-1},\ a_2^{-1}b_2^{-1}a_1^{-1}b_1\} \\
\cup &\{a_1b_3a_2^{-1}b_1,\ a_2^{-1}b_1a_1b_3,\ a_1^{-1}b_1^{-1}a_2b_3^{-1},\ a_2b_3^{-1}a_1^{-1}b_1^{-1}\} \\
\cup &\{a_1b_3^{-1}a_1b_2^{-1},\ a_1b_2^{-1}a_1b_3^{-1},\ a_1^{-1}b_2a_1^{-1}b_3,\ a_1^{-1}b_3a_1^{-1}b_2\} \\
\cup &\{a_1b_1^{-1}a_2^{-1}b_3,\ a_2^{-1}b_3a_1b_1^{-1},\ a_1^{-1}b_3^{-1}a_2b_1,\ a_2b_1a_1^{-1}b_3^{-1}\} \\
\cup &\{a_2b_3a_2b_2^{-1},\ a_2b_2^{-1}a_2b_3,\ a_2^{-1}b_2a_2^{-1}b_3^{-1},\ a_2^{-1}b_3^{-1}a_2^{-1}b_2\}.
\end{aligned}$$

In Figure 13, we can recognize a finite part of a periodic valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying (AC), with periods $(1,1)$, $(-2,2)$, $(4,0)$, $(0,4)$, and commuting elements $a_1a_2a_1a_2^{-1}$, $b_2^{-1}b_1^{-1}b_3^{-1}b_1$, generating $\mathbb{Z}^2 \cong \langle a_1a_2a_1a_2^{-1}, b_2^{-1}b_1^{-1}b_3^{-1}b_1 \rangle < \Gamma$. Note that they correspond to the commuting quaternions $5+4i+6j-2k$ and $-11-12i-18j+6k$ of norm 3^4 and 5^4 respectively.

However, recall that $\langle a_1, b_1 \rangle$ is an anti-torus in Γ, in particular there are also valid non-periodic tilings of the plane using the tile set $T(\Gamma)$. See Figure 14 for an illustration of a finite part of the non-periodic valid tiling determined by $\langle a_1, b_1 \rangle$. Note that all 24 squares of $T(\Gamma)$ appear in this picture.

Corollary 75. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n) \rangle$ *be a* $(2m, 2n)$*-group. Then there are always elements* $a \in \langle a_1, \ldots, a_m \rangle \setminus \{1\}$ *and* $b \in \langle b_1, \ldots, b_n \rangle \setminus \{1\}$ *such that* $\langle a, b \rangle$ *is no anti-torus.*

Proof. This follows directly from Proposition 74(3). □

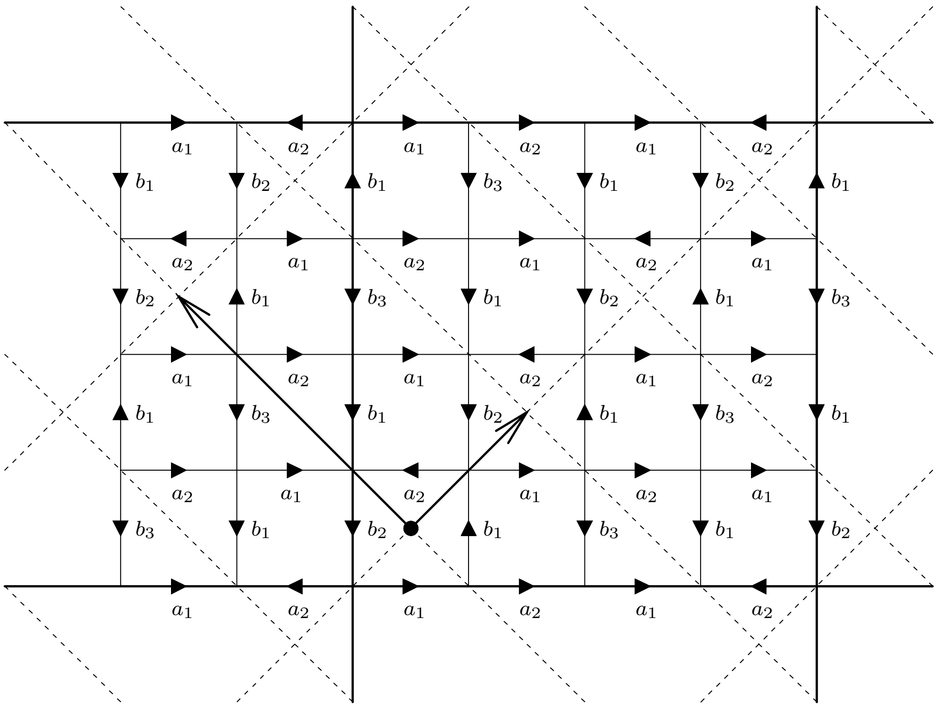


Figure 13: Illustration of Proposition 74 taking Example 52 and $t = a_1b_1a_2b_2$

7 Some 4-vertex square complexes

7.1 A very small candidate for simplicity

A $(2m, 2n)$-group Γ is never simple, since Γ_0 is a normal subgroup of index 4. However, we have conjectured Γ_0 to be a simple group in Example 1, 6, 7, 9, 10 and 11, and proved it to be simple in Example 14. The corresponding square complex X_0 has 4 vertices and $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$ as universal covering space. In this section, we construct a 4-vertex square complex Y, which is not a 4-fold covering of a $(2m, 2n)$-complex. Its universal covering space $\tilde{Y}$ is $\mathcal{T}_3 \times \mathcal{T}_4$. Observe that due to this more general construction, the degrees of the regular trees in $\tilde{Y}$ are not necessarily even. As a consequence, the number of geometric squares in Y is only 12 (compared to the 36 geometric squares of X_0 in Example 1 or the 100 geometric squares of X_0 in Example 14) and we get therefore relatively short presentations of $\pi_1 Y$. The construction of Y is done in such a way that Y is irreducible, all the “local groups” are at least 2-transitive and $\pi_1 Y$ is perfect. This seems to give some reasons to hope that $\pi_1 Y$ is a simple group. For the vertices α, β, γ, δ of Y, we denote the local groups by $P_h^{(k)}(\alpha)$, $P_v^{(k)}(\alpha)$, $P_h^{(k)}(\beta)$, $\ldots$, where $k \in \mathbb{N}$.

Example 54. *Let Y be the 4-vertex complex illustrated in Figure 15.*

Theorem 54. *Let Y be the 2-dimensional cell complex of Figure 15 with four vertices α, β, γ and δ. Then*

(1) *$Lk(\alpha) \cong Lk(\beta) \cong Lk(\gamma) \cong Lk(\delta) \cong K_{3,4}$ (complete bipartite graph). $\tilde{Y} = \mathcal{T}_3 \times \mathcal{T}_4$.*

(2) *$P_h(\alpha) \cong P_h(\delta) \cong S_3$, $P_h(\beta) \cong P_h(\gamma) \cong S_3$. $P_v(\alpha) \cong P_v(\beta) \cong S_4$, $P_v(\gamma) \cong P_v(\delta) \cong S_4$.*

(3) *Y is irreducible.*

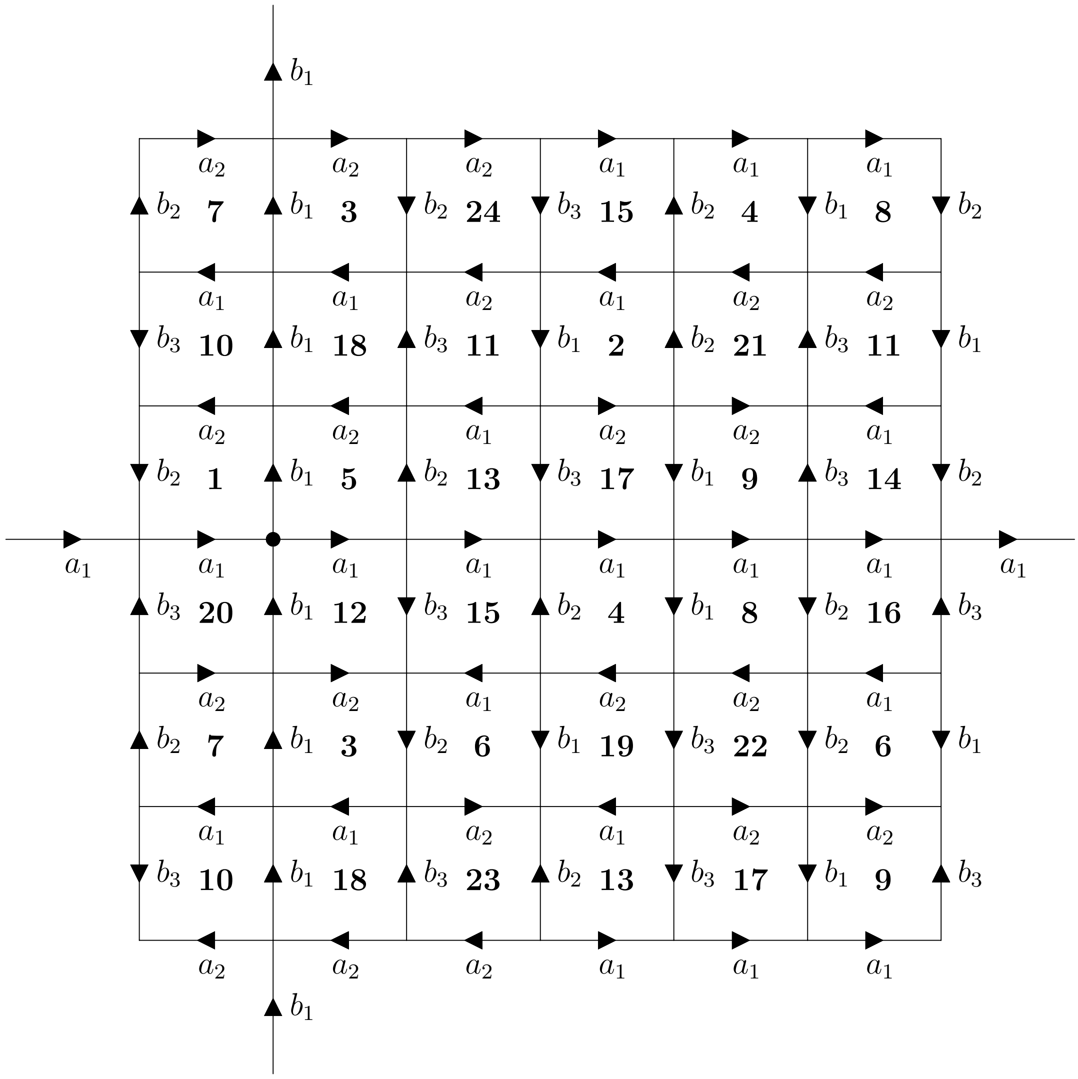


Figure 14: A non-periodic tiling of the plane determined by the anti-torus $\langle a_1, b_1 \rangle$ in $\Gamma_{3,5}$

(4) *$\pi_1 Y$ is a perfect group.*

(5) $F_3 *_{F_7} F_3 \cong \pi_1 Y \cong F_2 *_{F_5} F_2$.

Proof. (1) It can be directly read off from Figure 15.

(2) This follows from the definitions (see [16, Chapter 1]) and Figure 15. Note that for example $P_h(\alpha)$ and $P_h(\beta)$ could a priori be different, since α and β are not in the same connected component of the vertical 1-skeleton of Y. For an example where indeed $P_h(\alpha) \not\cong P_h(\beta)$, see Example 57.

(3) We compute

$$\left|P_v^{(2)}(\alpha)\right| = \left|P_v^{(2)}(\beta)\right| = \left|P_v^{(2)}(\gamma)\right| = \left|P_v^{(2)}(\delta)\right| = 24 \cdot 6^4.$$

The claim follows now from an obvious generalization of [16, Proposition 1.3] to the case where the horizontal 1-skeleton is not connected.

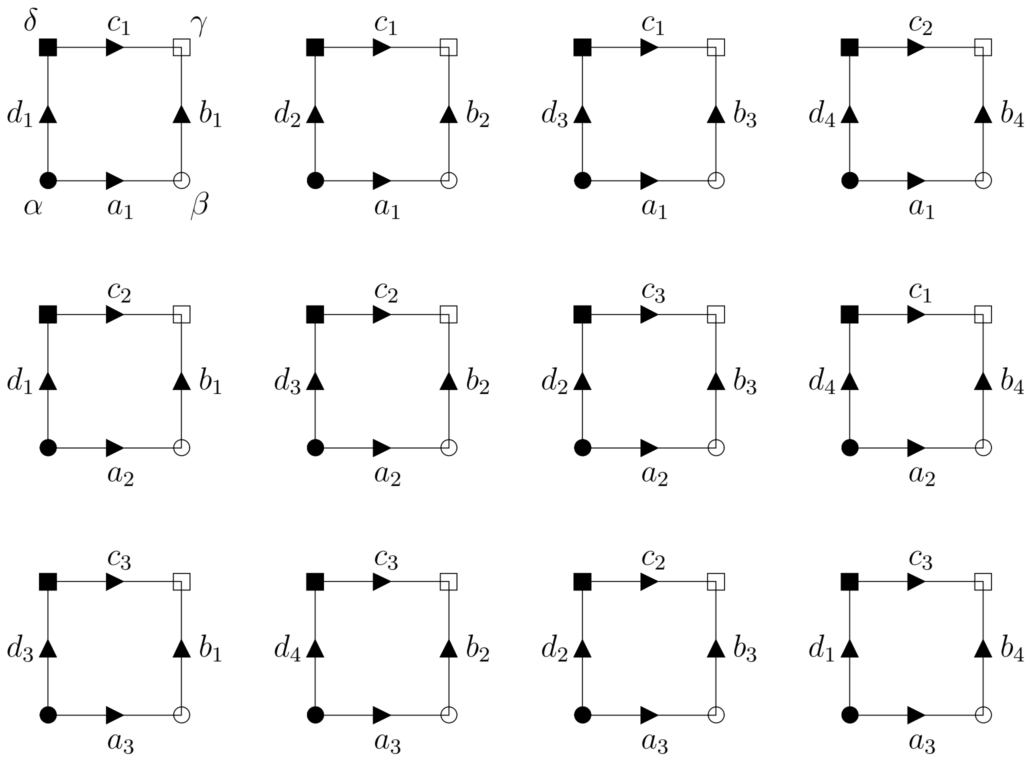


Figure 15: The 4-vertex square complex Y

(4) This follows directly from one of the presentations of $\pi_1 Y$ given in the proof of part (5).

(5) We give three "different" presentations of $\pi_1 Y$ and the corresponding isomorphisms. If we choose the vertex α as base point and the edges a_1, b_1, d_1 as "spanning tree" in the 1-skeleton of Y, we immediately get the following presentation of $\pi_1(Y, \alpha)$:

$$\begin{aligned}\pi_1(Y,\alpha) \cong \langle a_2,\, a_3,\, b_2,\, b_3,\, b_4, c_2,\, c_3,\, d_2,\, d_3,\, d_4 \mid &\\ b_2 = d_2,\ b_3 = d_3,\ b_4 = d_4 c_2,&\\ a_2 = c_2,\ a_2 b_2 = d_3 c_2,\ a_2 b_3 = d_2 c_3,\ a_2 b_4 = d_4,&\\ a_3 = d_3 c_3,\ a_3 b_2 = d_4 c_3,\ a_3 b_3 = d_2 c_2,\ a_3 b_4 = c_3\rangle,&\end{aligned}$$

i.e. after replacing c_2, d_2, d_3 by a_2, b_2, b_3 respectively

$$\begin{aligned}\pi_1(Y,\alpha) \cong \langle a_2,\, a_3, b_2,\, b_3,\, b_4,\, c_3,\, d_4 \mid &\\ b_4 = d_4 a_2,\ a_2 b_2 = b_3 a_2,\ a_2 b_3 = b_2 c_3,\ a_2 b_4 = d_4,&\\ a_3 = b_3 c_3,\ a_3 b_2 = d_4 c_3,\ a_3 b_3 = b_2 a_2,\ a_3 b_4 = c_3\rangle,&\end{aligned}$$

The two decompositions of $\pi_1 Y$ as amalgamated free products of free groups follow from [69, Theorem I.1.18].

$$\begin{aligned}F_3 *_{F_7} F_3 = \langle b_2,\, b_3,\, b_4,\, d_2,\, d_3,\, d_4 \mid d_2 = b_2,\ d_3 = b_3,\ d_4^2 = b_4^2,&\\ d_4 d_3 d_4^{-1} = b_4 b_2 b_4^{-1},&\\ d_4 d_2^2 d_4^{-1} = b_4 b_3 b_4^{-1} b_3 b_4^{-1},&\\ d_4 d_2^{-1} d_4 d_2 d_4^{-1} = b_4 b_3^{-1} b_2 b_4^{-1} b_3 b_4^{-1},&\\ d_4 d_2^{-1} d_3 d_2 d_4^{-1} = b_4 b_3^{-1} b_4^{-1} b_3 b_4^{-1}\rangle.&\end{aligned}$$

$$
\begin{aligned}
F_2 *_{F_5} F_2 = \langle a_2,\, a_3,\, c_2,\, c_3 \mid a_2 = c_2,\ & a_3^4 = c_3 c_2^{-1} c_3 c_2 c_3, \\
& a_3^{-1} a_2 a_3^{-2} = c_3^{-1} c_2 c_3^{-1} c_2 c_3^{-1}, \\
& a_3 a_2 a_3^{-1} = c_3 c_2^{-1} c_3^{-1},\ a_3^2 a_2 a_3 = c_3 c_2^{-1} c_3^3 \rangle.
\end{aligned}
$$

Isomorphisms between these three groups are given as follows:

$$
\begin{array}{rcccl}
\mathcal{T}_4 \curvearrowleft F_2 *_{F_5} F_2 & \overset{\cong}{\longleftrightarrow} & \pi_1(Y,\alpha) & \overset{\cong}{\longleftrightarrow} & F_3 *_{F_7} F_3 \curvearrowright \mathcal{T}_3 \\
 & & & & \\
a_2 & \longleftrightarrow & a_2 & \longleftrightarrow & d_4 b_4^{-1} \\
a_3 & \longleftrightarrow & a_3 & \longleftrightarrow & d_2 d_4^{-1} b_4 b_3^{-1} \\
a_2^{-1} a_3 c_3^{-1} c_2 & \longleftrightarrow & b_2 & \longleftrightarrow & b_2 \\
a_3 c_3^{-1} & \longleftrightarrow & b_3 & \longleftrightarrow & b_3 \\
a_3^{-1} c_3 & \longleftrightarrow & b_4 & \longleftrightarrow & b_4 \\
c_2 & \longleftrightarrow & c_2 & \longleftrightarrow & d_4 b_4^{-1} \\
c_3 & \longleftrightarrow & c_3 & \longleftrightarrow & d_2^{-1} d_4^{-1} b_4 b_3 \\
a_2^{-1} a_3 c_3^{-1} c_2 & \longleftrightarrow & d_2 & \longleftrightarrow & d_2 \\
a_3 c_3^{-1} & \longleftrightarrow & d_3 & \longleftrightarrow & d_3 \\
a_2 a_3^{-1} c_3 & \longleftrightarrow & d_4 & \longleftrightarrow & d_4 .
\end{array}
$$

□

Question 15. *Is it true that $\pi_1 Y$ does not have proper subgroups of finite index?*

Question 16. *Is $\pi_1 Y$ non-residually finite?*

Question 17. *Does every non-trivial normal subgroup of $\pi_1 Y$ have finite index?*

Question 18. *Is $\pi_1 Y$ simple?*

Remark. We have checked with GAP ([28]) that $\langle\langle w^k \rangle\rangle_{\pi_1 Y} = \pi_1 Y$, where $k = 1, \dots, 8$ and w is any generator of $\pi_1(Y, \alpha)$ in the first presentation given in the proof of Theorem 54(5).

7.2 More 4-vertex examples

We give several examples of a certain class of 4-vertex square complexes. In all examples, the complex will be denoted by Y. The 1-skeleton of Y and a typical geometric square of Y are illustrated in Figure 16, i.e. we always have four vertices α, β, γ, δ, horizontal edges a_1, a_2, a_3 (oriented from α to β), c_1, c_2, c_3 (oriented from δ to γ), and vertical edges $b_1, \dots, b_6$ (oriented from β to γ), $d_1, \dots, d_6$ (oriented from α to δ). Each of the 18 geometric squares is of the form $a_i b_j = d_l c_k$ (see the right hand side of Figure 16), and the universal covering space $\tilde{Y}$ is $\mathcal{T}_3 \times \mathcal{T}_6$. By construction of the 1-skeleton and the geometric squares of Y, we have for each $k \in \mathbb{N}$:

$$
P_h^{(k)}(\alpha) \cong P_h^{(k)}(\delta),\ P_h^{(k)}(\beta) \cong P_h^{(k)}(\gamma), \quad P_v^{(k)}(\alpha) \cong P_v^{(k)}(\beta),\ P_v^{(k)}(\gamma) \cong P_v^{(k)}(\delta).
$$

Example 55. *($(1, A_6)$, reducible)*
Let Y be given by its geometric squares

$$
\begin{array}{llllll}
a_1 b_1 = d_1 c_1, & a_1 b_2 = d_2 c_1, & a_1 b_3 = d_3 c_1, & a_1 b_4 = d_4 c_1, & a_1 b_5 = d_5 c_1, & a_1 b_6 = d_6 c_1, \\
a_2 b_1 = d_1 c_2, & a_2 b_2 = d_2 c_2, & a_2 b_3 = d_3 c_2, & a_2 b_4 = d_5 c_2, & a_2 b_5 = d_6 c_2, & a_2 b_6 = d_4 c_2, \\
a_3 b_1 = d_2 c_3, & a_3 b_2 = d_3 c_3, & a_3 b_3 = d_4 c_3, & a_3 b_4 = d_1 c_3, & a_3 b_5 = d_6 c_3, & a_3 b_6 = d_5 c_3.
\end{array}
$$

Then $P_h(\alpha) = 1$, $P_h(\beta) = 1$, $P_v(\alpha) = A_6$, $P_v(\gamma) = A_6$, $P_h^{(2)}(\alpha) = 1$, $P_h^{(2)}(\beta) = 1$, $P_v^{(2)}(\alpha) \cong A_6$, $P_v^{(2)}(\gamma) \cong A_6$.

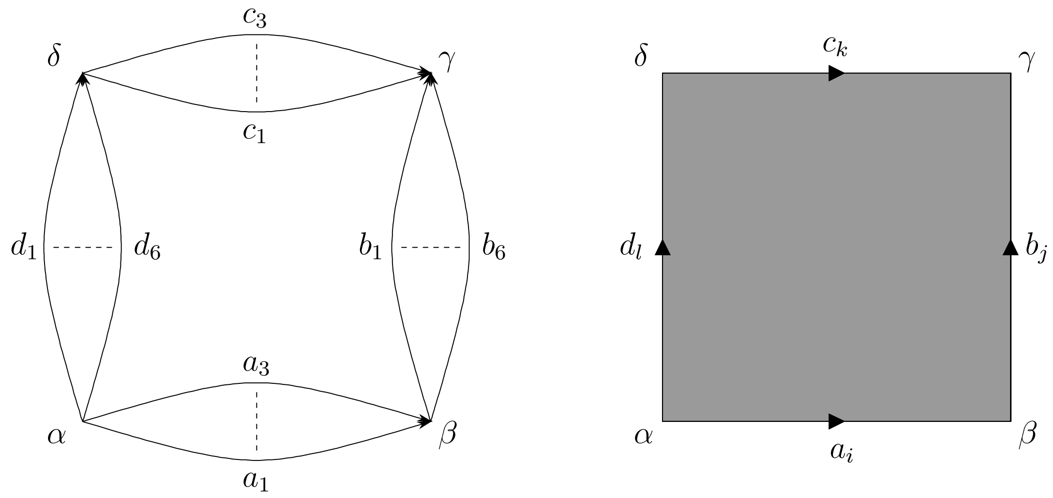


Figure 16: The 1-skeleton and a typical geometric square of Y

Example 56. *(*$(\mathbb{Z}_2, A_6)$*, irreducible)*
Let Y *be given by its geometric squares*

$$\begin{array}{llllll}
a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, & a_1b_4 = d_4c_1, & a_1b_5 = d_5c_1, & a_1b_6 = d_6c_1, \\
a_2b_1 = d_1c_2, & a_2b_2 = d_2c_2, & a_2b_3 = d_3c_2, & a_2b_4 = d_5c_2, & a_2b_5 = d_6c_2, & a_2b_6 = d_4c_3, \\
a_3b_1 = d_2c_3, & a_3b_2 = d_3c_3, & a_3b_3 = d_5c_3, & a_3b_4 = d_6c_3, & a_3b_5 = d_1c_3, & a_3b_6 = d_4c_2.
\end{array}$$

Then $P_h(\alpha) \cong \mathbb{Z}_2$*,* $P_h(\beta) \cong \mathbb{Z}_2$*,* $P_v(\alpha) = A_6$*,* $P_v(\gamma) = A_6$*,* $|P_h^{(2)}(\alpha)| = 4$*,* $|P_h^{(2)}(\beta)| = 4$*,* $|P_v^{(2)}(\alpha)| = 360 \cdot 60^6$*,* $|P_v^{(2)}(\gamma)| = 360 \cdot 60^6$*.*

Example 57. *(*$P_h(\alpha) \neq P_h(\beta)$*,* $|P_h^{(2)}(\alpha)| = |P_h(\alpha)|$*, irreducible)*
Let Y *be given by its geometric squares*

$$\begin{array}{llllll}
a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, & a_1b_4 = d_4c_1, & a_1b_5 = d_5c_2, & a_1b_6 = d_6c_3, \\
a_2b_1 = d_1c_2, & a_2b_2 = d_3c_2, & a_2b_3 = d_4c_2, & a_2b_4 = d_6c_2, & a_2b_5 = d_2c_3, & a_2b_6 = d_5c_1, \\
a_3b_1 = d_3c_3, & a_3b_2 = d_1c_3, & a_3b_3 = d_5c_3, & a_3b_4 = d_4c_3, & a_3b_5 = d_6c_1, & a_3b_6 = d_2c_2.
\end{array}$$

Then $|P_h(\alpha)| = 6$*,* $|P_h(\beta)| = 3$*,* $P_v(\alpha) = A_6$*,* $P_v(\gamma) = A_6$*,* $|P_h^{(2)}(\alpha)| = 6$*,* $|P_h^{(2)}(\beta)| = 24$*,* $|P_v^{(2)}(\alpha)| = 360 \cdot 60^6$*,* $|P_v^{(2)}(\gamma)| = 360 \cdot 60^6$*.*

Example 58. *(*$P_h(\alpha) \neq P_h(\beta)$*,* $P_v(\alpha) \neq P_v(\gamma)$*)*
Let Y *be given by its geometric squares*

$$\begin{array}{llllll}
a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, & a_1b_4 = d_4c_2, & a_1b_5 = d_5c_2, & a_1b_6 = d_6c_3, \\
a_2b_1 = d_1c_2, & a_2b_2 = d_3c_2, & a_2b_3 = d_4c_3, & a_2b_4 = d_5c_3, & a_2b_5 = d_6c_1, & a_2b_6 = d_2c_2, \\
a_3b_1 = d_2c_3, & a_3b_2 = d_3c_3, & a_3b_3 = d_6c_2, & a_3b_4 = d_4c_1, & a_3b_5 = d_1c_3, & a_3b_6 = d_5c_1.
\end{array}$$

Then $|P_h(\alpha)| = 3$*,* $|P_h(\beta)| = 6$*,* $|P_v(\alpha)| = 360$*,* $|P_v(\gamma)| = 120$*.*

A Supplement to Section 2

A.1 More (A_6, P_v)-groups

Example 59. $(A_6, \mathrm{PSL}_2(5))$

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_1b_2^{-1}, \\ \\ a_1b_3^{-1}a_3^{-1}b_2, & a_2b_1a_3b_2^{-1}, & a_2b_2a_3b_2, \\ \\ a_2b_3a_3b_1^{-1}, & a_2b_3^{-1}a_3b_3^{-1}, & a_2b_1^{-1}a_3b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,4)(3,5), \\ \rho_v(b_2) &= (1,6,5,3)(2,4), \\ \rho_v(b_3) &= (1,2,4,6)(3,5), \\ \\ \rho_h(a_1) &= (2,3)(4,5), \\ \rho_h(a_2) &= (1,3,4,5,2), \\ \rho_h(a_3) &= (2,3,4,6,5). \end{aligned}$$

Example 60. $(A_6, \mathrm{PGL}_2(5))$

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2, \\ \\ a_1b_3^{-1}a_2b_2^{-1}, & a_2b_1a_3^{-1}b_1^{-1}, & a_2b_2a_3^{-1}b_1, \\ \\ a_2b_3a_3b_3, & a_2b_1^{-1}a_3^{-1}b_2, & a_3b_2a_3b_3^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (1,5,4,3,2), \\ \rho_v(b_3) &= (2,6,5,3,4), \\ \\ \rho_h(a_1) &= (2,4,5,3), \\ \rho_h(a_2) &= (2,4,3,5,6), \\ \rho_h(a_3) &= (1,5,4,3,2). \end{aligned}$$

Example 61. (A_6, S_6)

$$R(3,3) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_2^{-1}b_3^{-1}, \\ & & \\ a_1b_3^{-1}a_3^{-1}b_3, & a_2b_1a_2^{-1}b_2^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, \\ & & \\ a_2b_3a_3^{-1}b_1, & a_2b_2^{-1}a_3b_1^{-1}, & a_3b_1a_3b_2 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,4,3), \\ \rho_v(b_2) &= (3,5,4), \\ \rho_v(b_3) &= (1,2,3)(4,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (), \\ \rho_h(a_2) &= (1,5,6,3,2), \\ \rho_h(a_3) &= (1,4,5)(2,6). \end{aligned}$$

Example 62. $(A_6, \mathrm{AGL}_1(8))$

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_4^{-1}, \\ & & \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ & & \\ a_1b_2^{-1}a_2^{-1}b_1, & a_1b_1^{-1}a_2^{-1}b_3, & a_2b_3a_3^{-1}b_2, \\ & & \\ a_3b_1a_3^{-1}b_4^{-1}, & a_3b_2a_3b_3, & a_3b_4a_3^{-1}b_1^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,2)(5,6), \\ \rho_v(b_2) &= (1,4,3,2)(5,6), \\ \rho_v(b_3) &= (1,2)(3,6,5,4), \\ \rho_v(b_4) &= (1,2)(5,6), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,6,8,7,5,4,3), \\ \rho_h(a_2) &= (1,2,4,5,6,7,3), \\ \rho_h(a_3) &= (1,4)(2,6)(3,7)(5,8). \end{aligned}$$

Example 63. $(A_6, \mathrm{A\Gamma L}_1(8))$

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_3^{-1}, & a_1b_2a_2b_3, & a_1b_3a_2^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_1, & a_1b_1^{-1}a_2^{-1}b_1^{-1}, & a_2b_2^{-1}a_3^{-1}b_3^{-1}, \\ a_3b_1a_3^{-1}b_1^{-1}, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_2 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,2)(5,6), \\ \rho_v(b_2) &= (1,4,5,6,2), \\ \rho_v(b_3) &= (1,2,3,6,5), \\ \rho_v(b_4) &= (1,2)(5,6), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,8,7,5,4,3)(2,6), \\ \rho_h(a_2) &= (1,2,4,5,6,8)(3,7), \\ \rho_h(a_3) &= (2,5,6)(3,7,4). \end{aligned}$$

Example 64. $(A_6, \mathrm{PSL}_2(7))$

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_1, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_4^{-1}, & a_1b_1^{-1}a_2^{-1}b_3, & a_2b_3a_3^{-1}b_2, \\ a_3b_1a_3^{-1}b_4, & a_3b_2a_3b_3, & a_3b_4a_3^{-1}b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,2)(5,6), \\ \rho_v(b_2) &= (1,4,3,2)(5,6), \\ \rho_v(b_3) &= (1,2)(3,6,5,4), \\ \rho_v(b_4) &= (1,2)(5,6), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,6,8)(4,7,5), \\ \rho_h(a_2) &= (1,7,3)(2,4,5), \\ \rho_h(a_3) &= (1,5)(2,6)(3,7)(4,8). \end{aligned}$$

Example 65. $(A_6, \mathrm{PGL}_2(7))$

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_3^{-1}, & a_1b_2a_2^{-1}b_1, & a_1b_3a_2^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_3, & a_1b_1^{-1}a_3^{-1}b_1^{-1}, & a_2b_1a_3^{-1}b_1, \\ a_2b_3a_3^{-1}b_2, & a_3b_2a_3b_3, & a_3b_4a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (1,3,2)(4,6,5), \\
\rho_v(b_2) &= (1,4,3,2)(5,6), \\
\rho_v(b_3) &= (1,2)(3,6,5,4), \\
\rho_v(b_4) &= (1,2)(5,6),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,8,2,6,7,5,4,3), \\
\rho_h(a_2) &= (1,7,3,2,4,5,6,8), \\
\rho_h(a_3) &= (1,8)(2,6)(3,7).
\end{aligned}$$

Example 66. (A_6, A_8)

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_1^{-1}b_3^{-1}, \\ a_1b_4a_2^{-1}b_4^{-1}, & a_1b_4^{-1}a_2^{-1}b_4, & a_2b_1a_3^{-1}b_2^{-1}, \\ a_2b_2a_3^{-1}b_2, & a_2b_3a_3b_1, & a_2b_3^{-1}a_2b_1^{-1}, \\ a_2b_2^{-1}a_3^{-1}b_3, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_1 \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (2,5,4), \\
\rho_v(b_2) &= (2,3)(4,5), \\
\rho_v(b_3) &= (2,5,3), \\
\rho_v(b_4) &= (1,2)(5,6),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (), \\
\rho_h(a_2) &= (1,6,7,2)(3,8), \\
\rho_h(a_3) &= (1,5,6)(2,7,8,4,3).
\end{aligned}$$

Example 67. (A_6, S_8)

$$R(3,4) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2, & a_1b_3a_1^{-1}b_3^{-1}, \\ a_1b_4a_2^{-1}b_4^{-1}, & a_1b_4^{-1}a_2^{-1}b_4, & a_2b_1a_3^{-1}b_2^{-1}, \\ a_2b_2a_3^{-1}b_2, & a_2b_3a_3b_1, & a_2b_3^{-1}a_2b_1^{-1}, \\ a_2b_2^{-1}a_3^{-1}b_3, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (2,3)(4,5), \\ \rho_v(b_3) &= (2,5,3), \\ \rho_v(b_4) &= (1,2)(5,6), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,7), \\ \rho_h(a_2) &= (1,6,7,2)(3,8), \\ \rho_h(a_3) &= (1,5,6)(2,7,8,4,3). \end{aligned}$$

Example 68. $(A_6, \mathrm{PSL}_2(9))$

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_3^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2^{-1}, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_1b_2^{-1}a_3^{-1}b_3, & a_2b_1a_2b_2^{-1}, & a_2b_2a_2b_3, \\ a_2b_5a_2b_1^{-1}, & a_2b_4^{-1}a_2b_3^{-1}, & a_3b_1a_3b_1^{-1}, \\ a_3b_3a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_5^{-1}, & a_3b_5a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5)(3,4), \\ \rho_v(b_2) &= (2,5)(4,6), \\ \rho_v(b_3) &= (1,3)(2,5), \\ \rho_v(b_4) &= (1,2,5), \\ \rho_v(b_5) &= (2,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,3)(4,5)(6,7)(8,9), \\ \rho_h(a_2) &= (1,5,4,8,2)(3,7,6,10,9), \\ \rho_h(a_3) &= (2,3)(4,5)(6,7)(8,9). \end{aligned}$$

Example 69. $(A_6, S_6 < S_{10})$

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_5, & a_1b_3a_1^{-1}b_3, \\ a_1b_4a_1^{-1}b_1, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_3, & a_2b_2a_2^{-1}b_2^{-1}, & a_2b_3a_3b_1, \\ a_2b_5a_2b_1^{-1}, & a_2b_4^{-1}a_2b_3^{-1}, & a_3b_2a_3^{-1}b_4^{-1}, \\ a_3b_3a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_1, & a_3b_5a_3^{-1}b_5 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (), \\ \rho_v(b_3) &= (2,5,3), \\ \rho_v(b_4) &= (1,2,5), \\ \rho_v(b_5) &= (2,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,7,6,2)(3,8)(4,5,9,10), \\ \rho_h(a_2) &= (1,5,4,8)(3,7,6,10), \\ \rho_h(a_3) &= (1,7,9,8)(2,3,10,4)(5,6). \end{aligned}$$

Example 70. $(A_6, \mathrm{PGL}_2(9))$

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_1^{-1}, & a_1b_3a_1^{-1}b_3, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_3, & a_2b_2a_2^{-1}b_2^{-1}, & a_2b_3a_3b_1, \\ a_2b_5a_2b_1^{-1}, & a_2b_4^{-1}a_2b_3^{-1}, & a_3b_2a_3^{-1}b_5, \\ a_3b_3a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_1, & a_3b_5a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (), \\ \rho_v(b_3) &= (2,5,3), \\ \rho_v(b_4) &= (1,2,5), \\ \rho_v(b_5) &= (2,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,2)(3,8)(4,5)(6,7)(9,10), \\ \rho_h(a_2) &= (1,5,4,8)(3,7,6,10), \\ \rho_h(a_3) &= (1,7,6,2,3,10,4,5,9,8). \end{aligned}$$

Example 71. (A_6, M_{10})

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2^{-1}, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_2b_3, & a_2b_5a_2b_1^{-1}, \\ a_2b_4^{-1}a_2b_3^{-1}, & a_2b_2^{-1}a_3b_1, & a_3b_2a_3^{-1}b_5^{-1}, \\ a_3b_3a_3b_3^{-1}, & a_3b_4a_3^{-1}b_1^{-1}, & a_3b_5a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (2,3,5), \\ \rho_v(b_3) &= (2,5)(3,4), \\ \rho_v(b_4) &= (1,2,5), \\ \rho_v(b_5) &= (2,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,3)(4,5)(6,7)(8,9), \\ \rho_h(a_2) &= (1,5,4,8,2)(3,7,6,10,9), \\ \rho_h(a_3) &= (1,4,5,2)(6,9,10,7). \end{aligned}$$

Example 72. $(A_6, \mathrm{P\Gamma L}_2(9))$

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2^{-1}, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_2b_3, & a_2b_5a_2b_1^{-1}, \\ a_2b_4^{-1}a_2b_3^{-1}, & a_2b_2^{-1}a_3b_1, & a_3b_2a_3^{-1}b_4, \\ a_3b_3a_3b_3^{-1}, & a_3b_4a_3^{-1}b_5, & a_3b_5a_3^{-1}b_1^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (2,3,5), \\ \rho_v(b_3) &= (2,5)(3,4), \\ \rho_v(b_4) &= (1,2,5), \\ \rho_v(b_5) &= (2,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,10)(2,3)(4,5)(6,7)(8,9), \\ \rho_h(a_2) &= (1,5,4,8,2)(3,7,6,10,9), \\ \rho_h(a_3) &= (1,5,7,2)(4,9,10,6). \end{aligned}$$

Example 73. (A_6, A_{10})

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_4^{-1}, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_4a_1^{-1}b_2^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_3^{-1}a_2^{-1}b_3, \\ a_2b_1a_3^{-1}b_1^{-1}, & a_2b_2a_3^{-1}b_2, & a_2b_4a_3^{-1}b_5^{-1}, \\ a_2b_5a_2b_4^{-1}, & a_2b_5^{-1}a_3b_4, & a_2b_2^{-1}a_3^{-1}b_1, \\ a_2b_1^{-1}a_3^{-1}b_2^{-1}, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_5a_3^{-1}b_3^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (2,3)(4,5), \\ \rho_v(b_3) &= (1,2)(5,6), \\ \rho_v(b_4) &= (2,5,4), \\ \rho_v(b_5) &= (2,3,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,4)(7,9), \\ \rho_h(a_2) &= (2,10,9)(4,5)(6,7), \\ \rho_h(a_3) &= (1,2,9)(3,5,4)(6,7,8). \end{aligned}$$

Example 74. (A_6, S_{10})

$$R(3,5) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_4, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_4a_1^{-1}b_2^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_3^{-1}a_2^{-1}b_3, \\ a_2b_1a_3^{-1}b_1^{-1}, & a_2b_2a_3^{-1}b_2, & a_2b_4a_3^{-1}b_5^{-1}, \\ a_2b_5a_2b_4^{-1}, & a_2b_5^{-1}a_3b_4, & a_2b_2^{-1}a_3^{-1}b_1, \\ a_2b_1^{-1}a_3^{-1}b_2^{-1}, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_5a_3^{-1}b_3^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (2,3)(4,5), \\ \rho_v(b_3) &= (1,2)(5,6), \\ \rho_v(b_4) &= (2,5,4), \\ \rho_v(b_5) &= (2,3,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,4,9,7), \\ \rho_h(a_2) &= (2,10,9)(4,5)(6,7), \\ \rho_h(a_3) &= (1,2,9)(3,5,4)(6,7,8). \end{aligned}$$

Example 75. $(A_6, \mathrm{PSL}_2(11))$

$$R(3,6) := \left\{ \begin{array}{lll} a_1b_1a_3^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_1^{-1}a_2b_2, & a_2b_1a_2b_3^{-1}, & a_2b_3a_2b_5^{-1}, \\ a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_1^{-1}a_3b_2, & a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_6, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,6,4,3,5), \\ \rho_v(b_2) &= (1,3,4,2,5), \\ \rho_v(b_3) &= (2,5)(3,4), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (2,5)(3,4), \\ \rho_v(b_6) &= (2,5)(3,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12), \\ \rho_h(a_2) &= (1,2,7,5,3)(6,11,12,10,8), \\ \rho_h(a_3) &= (1,2,7,5,3)(6,11,12,10,8). \end{aligned}$$

Example 76. $(A_6, \mathrm{PGL}_2(11))$

$$R(3,6) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_4^{-1}, & a_1b_3a_2^{-1}b_1, \\ a_1b_4a_1^{-1}b_6^{-1}, & a_1b_5a_1^{-1}b_3^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_3^{-1}a_3b_1^{-1}, & a_2b_1a_2b_2, & a_2b_3a_2b_5^{-1}, \\ a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_1^{-1}a_3b_3^{-1}, & a_3b_1a_3b_2, & a_3b_3a_3b_5^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_6, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,4,3,5,2), \\ \rho_v(b_2) &= (2,5)(3,4), \\ \rho_v(b_3) &= (2,4,3,6,5), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (2,5)(3,4), \\ \rho_v(b_6) &= (2,5)(3,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,10,8,7,9,11,12,3,5,6,4,2), \\ \rho_h(a_2) &= (1,10,8,6,11)(2,7,5,3,12), \\ \rho_h(a_3) &= (1,10,8,6,11)(2,7,5,3,12). \end{aligned}$$

Example 77. (A_6, A_{12})

$$R(3,6) := \left\{ \begin{array}{lll} a_1b_1a_3^{-1}b_1^{-1}, & a_1b_2a_3^{-1}b_1, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_2^{-1}a_3^{-1}b_2^{-1}, & a_1b_1^{-1}a_2b_2, & a_2b_1a_2b_3^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3^{-1}b_2, & a_3b_3a_3b_3^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_5^{-1}, & a_3b_6a_3b_6^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,3)(2,6,4,5), \\ \rho_v(b_2) &= (1,3,2,5)(4,6), \\ \rho_v(b_3) &= (2,5)(3,4), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (2,5)(3,4), \\ \rho_v(b_6) &= (2,5)(3,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,11,12)(3,4)(5,6)(7,8)(9,10), \\ \rho_h(a_2) &= (1,2,7,5,3)(6,11,12,10,8), \\ \rho_h(a_3) &= (1,11,2). \end{aligned}$$

Example 78. (A_6, S_{12})

$$R(3,6) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_3^{-1}, & a_2b_3a_2b_5^{-1}, \\ a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_1^{-1}a_3b_2, & a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_6, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (1,4,3,5,2), \\
\rho_v(b_2) &= (2,5,6,3,4), \\
\rho_v(b_3) &= (2,5)(3,4), \\
\rho_v(b_4) &= (), \\
\rho_v(b_5) &= (2,5)(3,4), \\
\rho_v(b_6) &= (2,5)(3,4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,2,12,11)(3,4)(5,6)(7,8)(9,10), \\
\rho_h(a_2) &= (1,2,7,5,3)(6,11,12,10,8), \\
\rho_h(a_3) &= (1,2,7,5,3)(6,11,12,10,8).
\end{aligned}$$

Example 79. $(A_6, \mathrm{PSL}_2(13))$

$$R(3,7) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_7a_1^{-1}b_7^{-1}, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_7^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_3^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,4,3,5,2), \\ \rho_v(b_2) &= (2,5,6,3,4), \\ \rho_v(b_3) &= (2,5)(3,4), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (2,5)(3,4), \\ \rho_v(b_6) &= (2,5)(3,4), \\ \rho_v(b_7) &= (2,5)(3,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,2)(3,4)(5,6)(9,10)(11,12)(13,14), \\ \rho_h(a_2) &= (1,2,9,5,3,7)(6,13,14,8,12,10), \\ \rho_h(a_3) &= (1,2,9,5,3,7)(6,13,14,8,12,10). \end{aligned}$$

Example 80. $(A_6, \mathrm{PGL}_2(13))$

$$R(3,7) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_7, \\ a_1b_4a_1^{-1}b_6^{-1}, & a_1b_5a_1^{-1}b_5, & a_1b_6a_1^{-1}b_4^{-1}, \\ a_1b_7a_1^{-1}b_3, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_7^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_3^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (1,4,3,5,2),\\
\rho_v(b_2) &= (2,5,6,3,4),\\
\rho_v(b_3) &= (2,5)(3,4),\\
\rho_v(b_4) &= (),\\
\rho_v(b_5) &= (2,5)(3,4),\\
\rho_v(b_6) &= (2,5)(3,4),\\
\rho_v(b_7) &= (2,5)(3,4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,2)(3,8)(4,6)(5,10)(7,12)(9,11)(13,14),\\
\rho_h(a_2) &= (1,2,9,5,3,7)(6,13,14,8,12,10),\\
\rho_h(a_3) &= (1,2,9,5,3,7)(6,13,14,8,12,10).
\end{aligned}$$

Example 81. (A_6, A_{14})

$$R(3,7) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_7a_1^{-1}b_7^{-1}, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_7^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,4,3,5,2), \\ \rho_v(b_2) &= (2,5,6,3,4), \\ \rho_v(b_3) &= (2,5)(3,4), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (2,5)(3,4), \\ \rho_v(b_6) &= (2,5)(3,4), \\ \rho_v(b_7) &= (2,5)(3,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1,2)(3,4)(5,6)(9,10)(11,12)(13,14), \\ \rho_h(a_2) &= (1,2,9,5,3,7)(6,13,14,8,12,10), \\ \rho_h(a_3) &= (1,2,9,5,3)(6,13,14,12,10). \end{aligned}$$

Example 82. (A_6, S_{14})

$$R(3,7) := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_7a_1^{-1}b_7, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_7^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (1,4,3,5,2), \\
\rho_v(b_2) &= (2,5,6,3,4), \\
\rho_v(b_3) &= (2,5)(3,4), \\
\rho_v(b_4) &= (), \\
\rho_v(b_5) &= (2,5)(3,4), \\
\rho_v(b_6) &= (2,5)(3,4), \\
\rho_v(b_7) &= (2,5)(3,4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14), \\
\rho_h(a_2) &= (1,2,9,5,3,7)(6,13,14,8,12,10), \\
\rho_h(a_3) &= (1,2,9,5,3)(6,13,14,12,10).
\end{aligned}$$

A.2 Amalgam decompositions of Example 1

We first give the vertical decomposition of Γ of Example 1:

$$\Gamma \cong F_3^{(v,b)} *_{F_{13}^{(v,b)} \cong F_{13}^{(v,s)}} F_7^{(v,s)}.$$

$F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle$, $F_7^{(v,s)} = \langle s_1, s_2, s_3, s_4, s_5, s_6, s_7 \rangle$. The inclusion $F_{13}^{(v,b)} \hookrightarrow F_3^{(v,b)}$ is given by the description of $F_{13}^{(v,b)}$ as a subgroup of $F_3^{(v,b)}$ of index 6:

$$\begin{aligned} F_{13}^{(v,b)} = \langle & b_1,\ b_3,\ b_2 b_3^{-1} b_2,\ b_2^{-1} b_3^{-1} b_2^2,\ b_2^{-1} b_1 b_2^2,\ b_2^{-1} b_1^{-1} b_2^2, \\ & b_2 b_1^{-2} b_2^{-1},\ b_2 b_3 b_1^{-1} b_2^{-1},\ b_2^2 b_1^{-1} b_2^{-1},\ b_2^{-3} b_1^{-1} b_2^{-1}, \\ & b_2 b_1 b_3^2 b_2^2,\ b_2^{-2} b_3^{-1} b_1 b_3 b_2^2,\ b_2^{-2} b_3^{-1} b_2 b_3 b_2^2 \rangle, \end{aligned}$$

the inclusion $F_{13}^{(v,s)} \hookrightarrow F_7^{(v,s)}$ by

$$\begin{aligned} F_{13}^{(v,s)} = \langle & s_1,\ s_2,\ s_6,\ s_4^{-1} s_3,\ s_5^{-1} s_3,\ s_7^{-1} s_3,\ s_7 s_3^{-1},\ s_5 s_3^{-1}, \\ & s_4 s_3^{-1},\ s_3^{-1} s_6 s_3^{-1},\ s_3^2,\ s_3^{-1} s_1 s_3,\ s_3^{-1} s_2 s_3 \rangle. \end{aligned}$$

The identification

$$\begin{array}{rcl} F_{13}^{(v,b)} & \stackrel{\cong}{\longleftrightarrow} & F_{13}^{(v,s)} \\ b_1 & \longleftrightarrow & s_1 \\ b_3 & \longleftrightarrow & s_2 \\ b_2 b_3^{-1} b_2 & \longleftrightarrow & s_6 \\ b_2^{-1} b_3^{-1} b_2^2 & \longleftrightarrow & s_4^{-1} s_3 \\ b_2^{-1} b_1 b_2^2 & \longleftrightarrow & s_5^{-1} s_3 \\ b_2^{-1} b_1^{-1} b_2^2 & \longleftrightarrow & s_7^{-1} s_3 \\ b_2 b_1^{-2} b_2^{-1} & \longleftrightarrow & s_7 s_3^{-1} \\ b_2 b_3 b_1^{-1} b_2^{-1} & \longleftrightarrow & s_5 s_3^{-1} \\ b_2^2 b_1^{-1} b_2^{-1} & \longleftrightarrow & s_4 s_3^{-1} \\ b_2^{-3} b_1^{-1} b_2^{-1} & \longleftrightarrow & s_3^{-1} s_6 s_3^{-1} \\ b_2 b_1 b_3^2 b_2^2 & \longleftrightarrow & s_3^2 \\ b_2^{-2} b_3^{-1} b_1 b_3 b_2^2 & \longleftrightarrow & s_3^{-1} s_1 s_3 \\ b_2^{-2} b_3^{-1} b_2 b_3 b_2^2 & \longleftrightarrow & s_3^{-1} s_2 s_3 \end{array}$$

leads to the following finite presentation of Γ:

$$\text{generators}(\Gamma) = \{b_1, b_2, b_3, s_1, s_2, s_3, s_4, s_5, s_6, s_7\},$$

$$\text{relations}(\Gamma) = \left\{ \begin{array}{rcl} b_1 & = & s_1, \\ b_3 & = & s_2, \\ b_2 b_3^{-1} b_2 & = & s_6, \\ b_2^{-1} b_3^{-1} b_2^2 & = & s_4^{-1} s_3, \\ b_2^{-1} b_1 b_2^2 & = & s_5^{-1} s_3, \\ b_2^{-1} b_1^{-1} b_2^2 & = & s_7^{-1} s_3, \\ b_2 b_1^{-2} b_2^{-1} & = & s_7 s_3^{-1}, \\ b_2 b_3 b_1^{-1} b_2^{-1} & = & s_5 s_3^{-1}, \\ b_2^2 b_1^{-1} b_2^{-1} & = & s_4 s_3^{-1}, \\ b_2^{-3} b_1^{-1} b_2^{-1} & = & s_3^{-1} s_6 s_3^{-1}, \\ b_2 b_1 b_3^2 b_2^2 & = & s_3^2, \\ b_2^{-2} b_3^{-1} b_1 b_3 b_2^2 & = & s_3^{-1} s_1 s_3, \\ b_2^{-2} b_3^{-1} b_2 b_3 b_2^2 & = & s_3^{-1} s_2 s_3 \end{array} \right\}.$$

In a similar way, we can describe the horizontal decomposition of

$$\Gamma \cong F_3^{(h,a)} *_{F_{13}^{(h,a)} \cong F_{13}^{(h,u)}} F_7^{(h,u)}.$$

$$\text{generators}(\Gamma) = \{a_1, a_2, a_3, u_1, u_2, u_3, u_4, u_5, u_6, u_7\},$$

$$\text{relations}(\Gamma) = \left\{ \begin{array}{rcl} a_1 & = & u_1, \\ a_3^4 & = & u_5 u_7, \\ a_2 a_3^{-3} & = & u_7 u_5^{-1}, \\ a_3^3 a_1 a_3^{-3} & = & u_5 u_1 u_5^{-1}, \\ a_3 a_1 a_3^{-2} & = & u_2 u_5^{-1}, \\ a_3 a_2 a_3^{-2} & = & u_3^{-1} u_5^{-1}, \\ a_3^2 a_1 a_3^{-1} & = & u_5 u_4, \\ a_3^2 a_2 a_3^{-1} & = & u_5 u_6, \\ a_3^3 a_2 a_1 a_2 & = & u_5 u_2, \\ a_3^3 a_2 a_3 a_2 & = & u_5 u_6^{-1}, \\ a_3^3 a_2^3 & = & u_5^2, \\ a_2^{-1} a_3 a_2^{-1} a_3^{-3} & = & u_4 u_5^{-1}, \\ a_2^{-1} a_1 a_2^{-1} a_3^{-3} & = & u_3 u_5^{-1} \end{array} \right\}.$$

We recall the relators $R(3,3)$ of Example 1:

$$R(3,3) := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, & a_1 b_3 a_2 b_2^{-1}, \\ \\ a_1 b_3^{-1} a_3^{-1} b_2, & a_2 b_1 a_3^{-1} b_2^{-1}, & a_2 b_2 a_3^{-1} b_3^{-1}, \\ \\ a_2 b_3 a_3^{-1} b_1, & a_2 b_3^{-1} a_3 b_2, & a_2 b_1^{-1} a_3^{-1} b_1^{-1} \end{array} \right\}.$$

Explicit isomorphisms between the three given presentations of Γ are:

$$\begin{array}{rcccl} F_3^{(v,b)} *_{F_{13}^{(v,b)} \cong F_{13}^{(v,s)}} F_7^{(v,s)} & \overset{\cong}{\longleftrightarrow} & \langle a_1, \dots, b_3 \mid R(3,3) \rangle & \overset{\cong}{\longleftrightarrow} & F_3^{(h,a)} *_{F_{13}^{(h,a)} \cong F_{13}^{(h,u)}} F_7^{(h,u)} \\ \\ s_3 b_2^{-2} b_3^{-1} & \longleftrightarrow & a_1 & \longleftrightarrow & a_1 = u_1 \\ b_3 b_2 s_4^{-1} b_2 & \longleftrightarrow & a_2 & \longleftrightarrow & a_2 \\ b_2 s_4^{-1} b_2^2 & \longleftrightarrow & a_3 & \longleftrightarrow & a_3 \\ s_1 = b_1 & \longleftrightarrow & b_1 & \longleftrightarrow & u_7^{-1} a_2 \\ b_2 & \longleftrightarrow & b_2 & \longleftrightarrow & a_2 u_5^{-1} a_3^2 \\ s_2 = b_3 & \longleftrightarrow & b_3 & \longleftrightarrow & a_2^2 u_5^{-1} a_3 \\ s_3 & \longleftrightarrow & a_1 b_3 b_2^2 = b_2 b_1 b_3 a_1^{-1} & & \\ s_4 & \longleftrightarrow & a_1 b_3^2 b_2 = b_2^2 b_3 a_1^{-1} & & \\ s_5 & \longleftrightarrow & a_1 b_3 b_1^{-1} b_2 = b_2 b_3^2 a_1^{-1} & & \\ s_6 & \longleftrightarrow & b_2 b_3^{-1} b_2 & & \\ s_7 & \longleftrightarrow & a_1 b_3 b_1 b_2 = b_2 b_1^{-1} b_3 a_1^{-1} & & \\ & & a_3 a_1 a_3 b_1^{-1} = b_1 a_2 a_1 a_2 & \longleftrightarrow & u_2 \\ & & a_2^{-1} a_1 a_2^{-1} b_1^{-1} = b_1 (a_3 a_2 a_3)^{-1} & \longleftrightarrow & u_3 \\ & & a_2^{-1} a_3 a_2^{-1} b_1^{-1} = b_1 a_3^{-1} a_1 a_3^{-1} & \longleftrightarrow & u_4 \\ & & a_3^3 b_1^{-1} = b_1 a_2^3 & \longleftrightarrow & u_5 \\ & & (b_1 a_2 a_3 a_2)^{-1} = b_1 a_3^{-1} a_2 a_3^{-1} & \longleftrightarrow & u_6 \\ & & a_2 b_1^{-1} = b_1 a_3 & \longleftrightarrow & u_7. \end{array}$$

With this identification, the abelianization map $\Gamma \to \Gamma^{ab} \cong \mathbb{Z}_2^2$ is now

$$\begin{aligned}
a_1, a_2, a_3 &\mapsto (1,0)\\
b_1, b_2, b_3 &\mapsto (0,1)\\
s_1, s_2, s_6 &\mapsto (0,1)\\
s_3, s_4, s_5, s_7 &\mapsto (1,1)\\
u_1 &\mapsto (1,0)\\
u_2, u_3, u_4, u_5, u_6, u_7 &\mapsto (1,1).
\end{aligned}$$

The vertical amalgam decomposition of Γ described above gives a natural action of Γ on the first barycentric subdivision $\mathcal{T}_6'$ of $\mathcal{T}_{2m} = \mathcal{T}_6$. See [66, Chapter 4] for the general theory about amalgams and their action on the corresponding tree. Let P be the vertex of $\mathcal{T}_6'$ stabilized by $F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle$. The local action of $\Gamma \cong \mathrm{pr}_1(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2m})$ on $S(x_h, 1)$ in $\mathcal{T}_6$, i.e. the homomorphism $\rho_v : \langle b_1, b_2, b_3 \rangle \twoheadrightarrow P_h < S_{2m}$ determined in the proof of Theorem 1(1), can be reconstructed by the action of $F_3^{(v,b)}$ on the set of edges of $\mathcal{T}_6'$ originating at P. These edges are labelled by right cosets $F_{13}^{(v,b)} g_i$, $i = 1, \ldots, 6$, $g_i \in F_3^{(v,b)}$, such that

$$F_3^{(v,b)} = \bigsqcup_{i=1}^{6} F_{13}^{(v,b)} g_i.$$

The group $F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle$ acts by right multiplication on the set of right cosets $\{F_{13}^{(v,b)} g_i\}_{i=1,\ldots,6}$. If we choose $g_1 = 1$, $g_2 = b_2 b_1 b_2$, $g_3 = (b_2 b_1)^2$, $g_4 = b_2 b_1$, $g_5 = b_2$, $g_6 = b_2 b_1 b_3$ and make the identification $F_{13}^{(v,b)} g_i \leftrightarrow i$ for $i = 1, \ldots, 6$, then we exactly get back our ρ_v:

$$\begin{aligned}
&\rho_v(b_1) = (2,3)(4,5),\\
&\rho_v(b_2) = (1,5,4,2,3),\\
&\rho_v(b_3) = (2,3,5,4,6), \text{ generating } P_h = A_6.
\end{aligned}$$

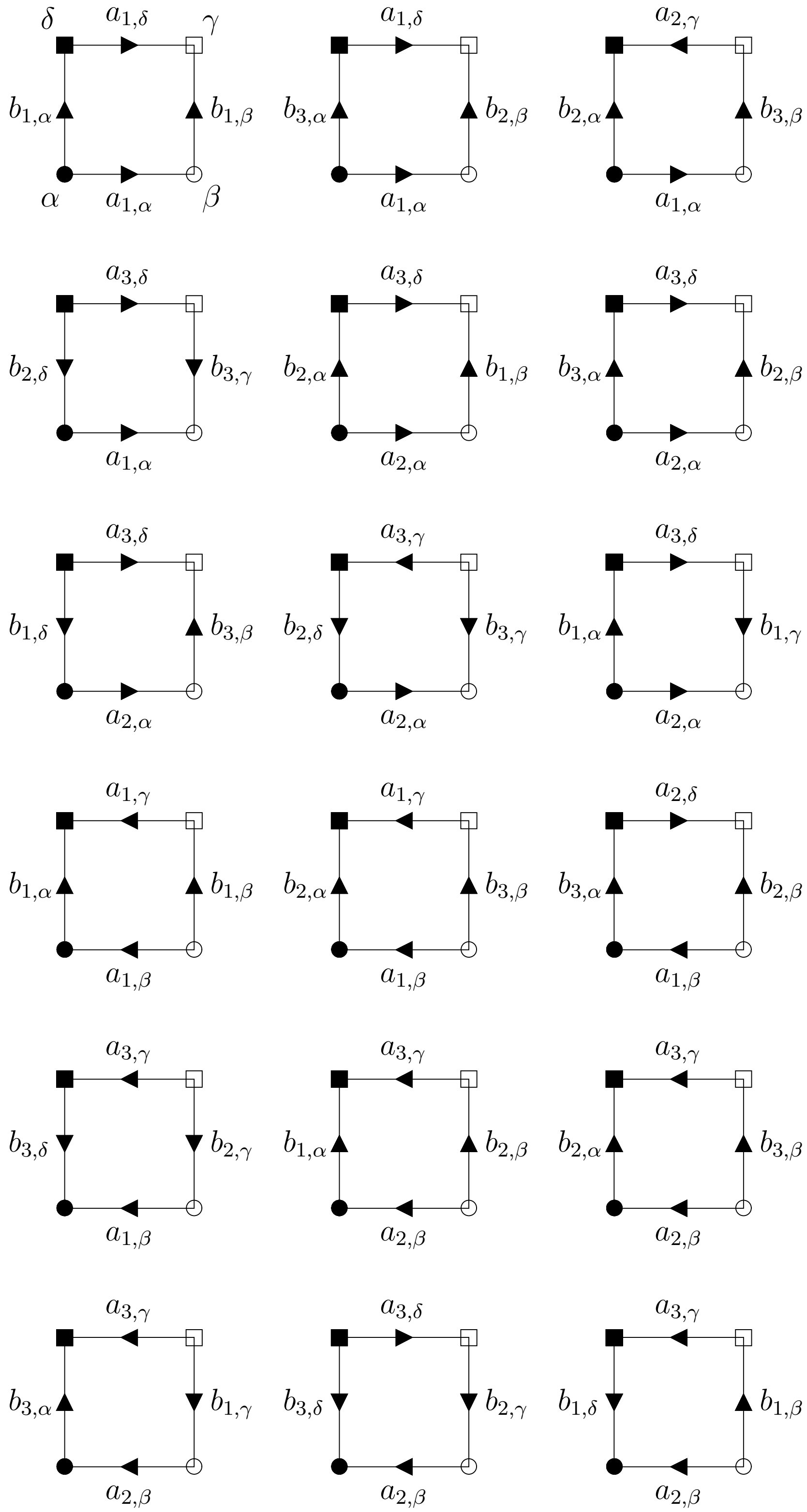
In the same way, we compute the action of $F_3^{(h,a)} = \langle a_1, a_2, a_3 \rangle$ by right multiplication on right cosets

$$F_3^{(h,a)} = F_{13}^{(h,a)} \sqcup F_{13}^{(h,a)} a_2^2 a_1 \sqcup F_{13}^{(h,a)} a_2^2 \sqcup F_{13}^{(h,a)} a_3 \sqcup F_{13}^{(h,a)} a_3 a_1 \sqcup F_{13}^{(h,a)} a_2$$

and recover $\rho_h : \langle a_1, a_2, a_3 \rangle \twoheadrightarrow P_v < S_{2n} = S_6$:

$$\begin{aligned}
&\rho_h(a_1) = (2,3)(4,5),\\
&\rho_h(a_2) = (1,6,3,2)(4,5),\\
&\rho_h(a_3) = (1,4,5,6)(2,3), \text{ generating } P_v = A_6.
\end{aligned}$$

The cell complex X_0 of Example 1 corresponding to Γ_0 is given by the following $4 \cdot 9 = 36$ geometric squares:



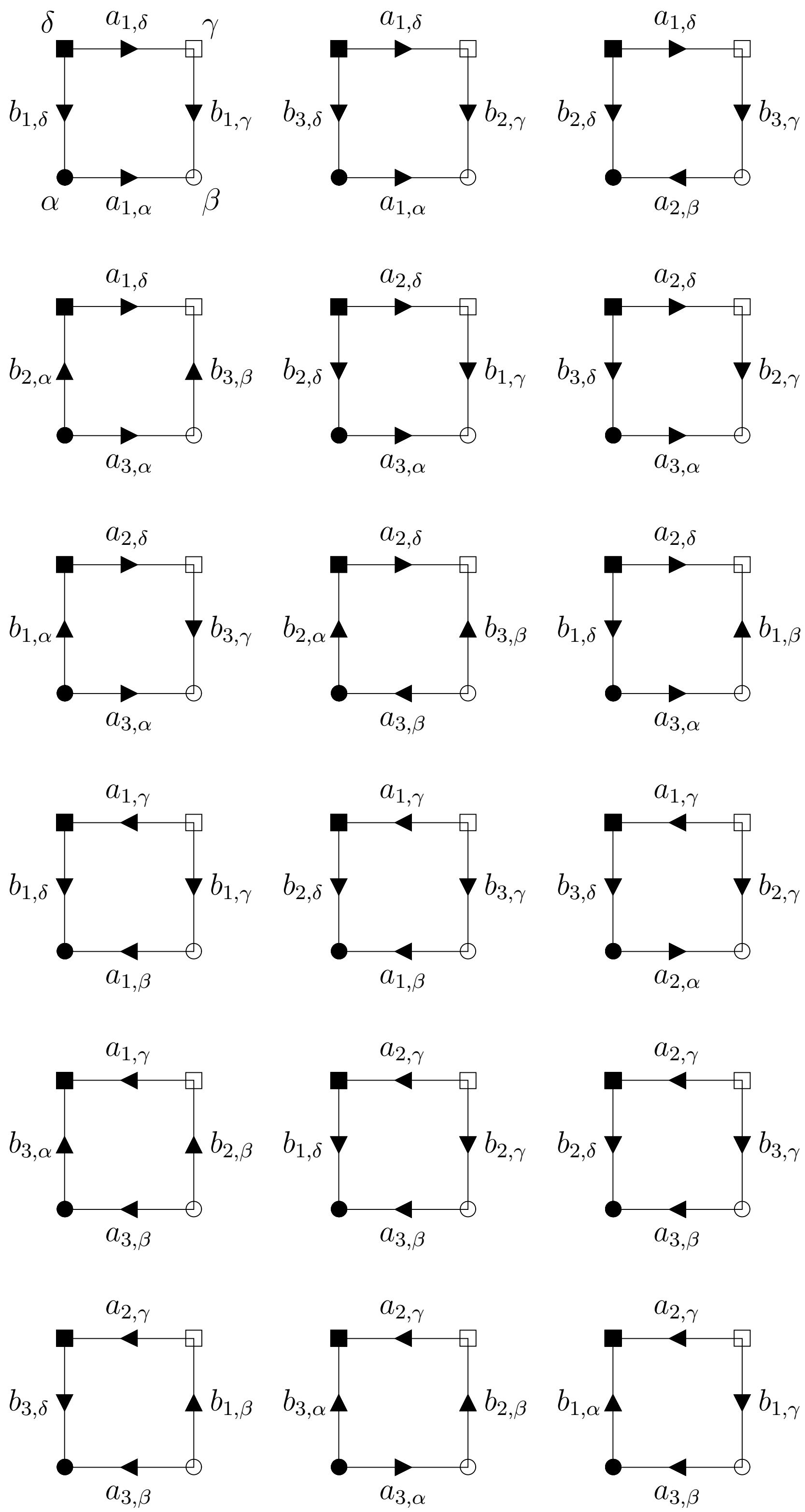


Figure 17: X_0 of Example 1

The decompositions of Γ_0 are:

$$F_5^{(v,r)} *_{F_{25}^{(v,r)} \cong F_{25}^{(v,q)}} F_5^{(v,q)} \cong \Gamma_0 \cong F_5^{(h,t)} *_{F_{25}^{(h,t)} \cong F_{25}^{(h,w)}} F_5^{(h,w)},$$

where $F_5^{(v,r)} = \langle r_1, r_2, r_3, r_4, r_5 \rangle$, $F_5^{(v,q)} = \langle q_1, q_2, q_3, q_4, q_5 \rangle$. The inclusion $F_{25}^{(v,r)} \hookrightarrow F_5^{(v,r)}$ is defined by

$$\begin{aligned} F_{25}^{(v,r)} = \langle & r_2,\, r_5,\, r_3,\, r_1 r_5 r_3^{-1} r_1^{-1},\, r_1 r_4 r_3^{-1} r_1^{-1},\, r_1 r_3 r_1^{-1},\, r_1^{-1} r_5 r_1, \\ & r_1^{-1} r_3 r_1,\, r_1^{-1} r_4 r_1,\, r_1^{-1} r_2 r_1^{-1},\, r_4^{-1} r_1^{-1} r_4,\, r_4^{-1} r_5 r_1 r_4, \\ & r_4^{-1} r_1^{-1} r_2 r_4,\, r_4 r_1 r_4^{-1},\, r_4 r_2 r_4^{-1},\, r_4 r_5 r_4^{-1},\, r_4 r_3^{-1} r_4, \\ & r_4 r_3 r_2 r_1,\, r_4 r_3 r_4 r_3^{-1} r_4^{-1},\, r_4 r_3 r_5 r_3^{-1} r_4^{-1},\, r_4 r_3 r_1 r_3^{-1} r_4^{-1}, \\ & r_4^2 r_1 r_4,\, r_1 r_3 r_1^2,\, r_1 r_3 r_2 r_3^{-1} r_4^{-1},\, r_4 r_3^2 r_1 r_4 \rangle \end{aligned}$$

and the other inclusion $F_{25}^{(v,q)} \hookrightarrow F_5^{(v,q)}$ by

$$\begin{aligned} F_{25}^{(v,q)} = \langle & q_1,\, q_5,\, q_4,\, q_2 q_4 q_2^{-1},\, q_2 q_3 q_2^{-1},\, q_2 q_5^{-1} q_2^{-1},\, q_2^{-1} q_3^{-1} q_2, \\ & q_2^{-1} q_3^{-1} q_4 q_2,\, q_2^{-1} q_3^{-1} q_5 q_2,\, q_2^{-1} q_1 q_2^{-1},\, q_3^{-1} q_5^{-1} q_3, \\ & q_3^{-1} q_2^{-1} q_3,\, q_3^{-1} q_1 q_3,\, q_3 q_2 q_1^{-1} q_3^{-1},\, q_3 q_5^{-1} q_1^{-1} q_3^{-1}, \\ & q_3 q_1 q_3^{-1},\, q_3 q_4^{-1} q_3,\, q_3 q_1 q_4 q_1 q_3 q_2,\, q_3 q_1 q_4 q_3 q_4^{-1} q_1^{-1} q_3^{-1}, \\ & q_3 q_1 q_4 q_5 q_4^{-1} q_1^{-1} q_3^{-1},\, q_3 q_1 q_4 q_2 q_4^{-1} q_1^{-1} q_3^{-1},\, q_3 q_1 q_3^2, \\ & q_2^2 q_3 q_2,\, q_2 q_1 q_4^{-1} q_1^{-1} q_3^{-1},\, q_3 q_1 q_4^2 q_3 \rangle. \end{aligned}$$

We obtain a finite presentation for the vertical decomposition of Γ_0:

$$\text{generators}(\Gamma_0) = \{r_1, r_2, r_3, r_4, r_5, q_1, q_2, q_3, q_4, q_5\},$$

$$\text{relations}(\Gamma_0) = \left\{ \begin{array}{rcl} r_2 & = & q_1, \\ r_5 & = & q_5, \\ r_3 & = & q_4, \\ r_1 r_5 r_3^{-1} r_1^{-1} & = & q_2 q_4 q_2^{-1}, \\ r_1 r_4 r_3^{-1} r_1^{-1} & = & q_2 q_3 q_2^{-1}, \\ r_1 r_3 r_1^{-1} & = & q_2 q_5^{-1} q_2^{-1}, \\ r_1^{-1} r_5 r_1 & = & q_2^{-1} q_3^{-1} q_2, \\ r_1^{-1} r_3 r_1 & = & q_2^{-1} q_3^{-1} q_4 q_2, \\ r_1^{-1} r_4 r_1 & = & q_2^{-1} q_3^{-1} q_5 q_2, \\ r_1^{-1} r_2 r_1^{-1} & = & q_2^{-1} q_1 q_2^{-1}, \\ r_4^{-1} r_1^{-1} r_4 & = & q_3^{-1} q_5^{-1} q_3, \\ r_4^{-1} r_5 r_1 r_4 & = & q_3^{-1} q_2^{-1} q_3, \\ r_4^{-1} r_1^{-1} r_2 r_4 & = & q_3^{-1} q_1 q_3, \\ r_4 r_1 r_4^{-1} & = & q_3 q_2 q_1^{-1} q_3^{-1}, \\ r_4 r_2 r_4^{-1} & = & q_3 q_5^{-1} q_1^{-1} q_3^{-1}, \\ r_4 r_5 r_4^{-1} & = & q_3 q_1 q_3^{-1}, \\ r_4 r_3^{-1} r_4 & = & q_3 q_4^{-1} q_3, \\ r_4 r_3 r_2 r_1 & = & q_3 q_1 q_4 q_1 q_3 q_2, \\ r_4 r_3 r_4 r_3^{-1} r_4^{-1} & = & q_3 q_1 q_4 q_3 q_4^{-1} q_1^{-1} q_3^{-1}, \\ r_4 r_3 r_5 r_3^{-1} r_4^{-1} & = & q_3 q_1 q_4 q_5 q_4^{-1} q_1^{-1} q_3^{-1}, \\ r_4 r_3 r_1 r_3^{-1} r_4^{-1} & = & q_3 q_1 q_4 q_2 q_4^{-1} q_1^{-1} q_3^{-1}, \\ r_4^2 r_1 r_4 & = & q_3 q_1 q_3^2, \\ r_1 r_3 r_1^2 & = & q_2^2 q_3 q_2, \\ r_1 r_3 r_2 r_3^{-1} r_4^{-1} & = & q_2 q_1 q_4^{-1} q_1^{-1} q_3^{-1}, \\ r_4 r_3^2 r_1 r_4 & = & q_3 q_1 q_4^2 q_3 \end{array} \right\}.$$

The horizontal decomposition of Γ_0 is:

$$\text{generators}(\Gamma_0) = \{w_1, w_2, w_3, w_4, w_5, t_1, t_2, t_3, t_4, t_5\},$$

$$\text{relations}(\Gamma_0) = \left\{ \begin{array}{rcl} w_1 w_5 &=& t_2 t_4, \\ w_1 w_4^2 &=& t_2 t_5^2, \\ w_3 &=& t_3, \\ w_1 w_3 w_1^{-1} &=& t_2 t_3 t_2^{-1}, \\ w_4 w_1^{-1} &=& t_5 t_2^{-1}, \\ w_1 w_2 &=& t_2 t_1, \\ w_4^{-1} w_1 w_4 &=& t_5^{-1} t_2 t_5, \\ w_4^{-1} w_3^{-1} w_4 &=& t_5^{-1} t_1 t_5, \\ w_4^{-1} w_5 w_4 &=& t_5^{-1} t_3 t_5, \\ w_4^{-1} w_2^{-1} w_4 &=& t_5^{-1} t_4 t_5, \\ w_1 w_2^{-2} &=& t_2 t_1^{-1} t_2 t_1^{-1}, \\ w_2 w_1^{-1} w_2 w_1^{-1} &=& t_1 t_5 t_1 t_2^{-1}, \\ w_1 w_2^{-1} w_4^{-1} w_2^{-1} &=& t_2 t_1^{-2}, \\ w_2 w_3 w_2 w_1^{-1} &=& t_1 t_3 t_1 t_2^{-1}, \\ w_2 w_5 w_2 w_1^{-1} &=& t_1 t_4 t_1 t_2^{-1}, \\ w_1 w_5^{-1} w_3 w_5^{-1} &=& t_2 t_4^{-1} t_5 t_4^{-1}, \\ w_5 w_4^{-1} w_5 w_1^{-1} &=& t_4 t_2 t_4 t_2^{-1}, \\ w_5^2 w_1^{-1} &=& t_4^2 t_2^{-1}, \\ w_5 w_2 w_5 w_1^{-1} &=& t_4 t_1 t_4 t_2^{-1}, \\ w_1 w_5^{-1} w_1^{-1} w_5^{-1} &=& t_2 t_4^{-1} t_3 t_4^{-1}, \\ w_1^{-1} w_5^{-1} w_1^{-2} &=& t_2^{-1} t_1 t_2^{-2}, \\ w_1^{-1} w_2 w_1^{-2} &=& t_2^{-3}, \\ w_1^2 w_4 w_1 &=& t_2^2 t_5 t_2, \\ w_1^2 w_3 w_1 &=& t_2^2 t_3 t_2, \\ w_1^3 &=& t_2^2 t_4 t_2 \end{array} \right\}.$$

Explicit isomorphisms between the two amalgams of Γ_0 and Γ_0 as a subgroup of Γ are given as follows:

$$\begin{array}{rccc}
 & & \Gamma & \\
 & & \vee & \\
F_5^{(v,r)} *_{F_{25}^{(v,r)} \cong F_{25}^{(v,q)}} F_5^{(v,q)} & \stackrel{\cong}{\longleftrightarrow} & \Gamma_0 & \stackrel{\cong}{\longleftrightarrow} & F_5^{(h,t)} *_{F_{25}^{(h,t)} \cong F_{25}^{(h,w)}} F_5^{(h,w)} \\
 & & & & \\
r_1 & \longleftrightarrow & b_2 b_1^{-1} & \longleftrightarrow & w_1 t_2^{-1} \\
r_2 = q_1 & \longleftrightarrow & b_3 b_1^{-1} & \longleftrightarrow & w_4 t_5^{-1} \\
r_3 = q_4 & \longleftrightarrow & b_1 b_3 & \longleftrightarrow & t_4^{-1} w_5 \\
r_4 & \longleftrightarrow & b_1 b_2 & \longleftrightarrow & t_1^{-1} w_2 w_4 \\
r_5 = q_5 & \longleftrightarrow & b_1^2 & \longleftrightarrow & t_5^{-1} w_4 \\
q_2 & \longleftrightarrow & a_1 a_3^{-1} b_2 b_1^{-1} = b_2 b_3 a_3^{-1} a_1^{-1} & \longleftrightarrow & w_2^{-1} w_1 t_2^{-1} \\
q_3 & \longleftrightarrow & a_1 a_2^{-1} b_2^2 = b_1 b_2 a_2^{-1} a_1^{-1} & \longleftrightarrow & t_2^{-1} t_1^{-1} w_2 w_4 \\
r_1 r_4 q_3^{-1} & \longleftrightarrow & a_2 a_1^{-1} & \longleftrightarrow & w_1 \\
r_1 q_2^{-1} & \longleftrightarrow & a_3 a_1^{-1} & \longleftrightarrow & w_2 \\
q_1^{-1} q_2^{-1} r_1 r_3 r_2 & \longleftrightarrow & a_1^2 & \longleftrightarrow & w_3 = t_3 \\
q_3^{-1} r_4 & \longleftrightarrow & a_1 a_2 & \longleftrightarrow & w_4 \\
q_2^{-1} r_1 r_3 & \longleftrightarrow & a_1 a_3 & \longleftrightarrow & w_5 \\
r_1 q_2^{-1} q_3^{-1} & \longleftrightarrow & a_3 a_2 b_2^{-1} b_1^{-1} = b_1^2 a_3 a_1^{-1} & \longleftrightarrow & t_1 \\
r_1 r_4 q_3^{-1} q_5^{-1} & \longleftrightarrow & a_2 a_1^{-1} b_1^{-2} = b_1 b_2^{-1} a_2 a_1^{-1} & \longleftrightarrow & t_2 \\
q_2^{-1} r_1 & \longleftrightarrow & a_1 a_3 b_3^{-1} b_1^{-1} = b_1^2 a_1 a_3 & \longleftrightarrow & t_4 \\
q_3^{-1} r_4 r_5^{-1} & \longleftrightarrow & a_1 a_2 b_1^{-2} = b_1 b_3^{-1} a_1 a_2 & \longleftrightarrow & t_5
\end{array}$$

A.3 Illustration of Theorem 1(7)

In the notation of the proof of [16, Proposition 6.1] we have $n = 0$, ${}^{(0)}X = (A_6, A_6)$-complex X of Example 1 and $k = \ell = 4$. Let $C_{k,\ell}$ be given by

$$\{a_4b_4a_5^{-1}b_5^{-1},\ a_4b_4^{-1}a_5^{-1}b_5,\ a_4b_5a_5^{-1}b_4^{-1},\ a_4b_5^{-1}a_5^{-1}b_4\}$$

and $C_{4,4}$ (a disjoint copy of $C_{k,\ell}$) be given by

$$\{a_6b_6a_7^{-1}b_7^{-1},\ a_6b_6^{-1}a_7^{-1}b_7,\ a_6b_7a_7^{-1}b_6^{-1},\ a_6b_7^{-1}a_7^{-1}b_6\}.$$

We choose ${}^{(0)}a := a_1$, ${}^{(0)}b := b_1$, $\widehat{a}_1 := a_4$, $\widehat{a}_2 := a_5$, $\widehat{b}_1 := b_4$, $\widehat{b}_2 := b_5$, $\widetilde{a}_1 := a_6$, $\widetilde{a}_2 := a_7$, $\widetilde{b}_1 := b_6$, $\widetilde{b}_2 := b_7$. The surgery described in the proof of [16, Proposition 6.1] leads to the irreducible (A_{14}, A_{14})-complex given by the following $R(7,7)$ (the relators of the embedded Example 1 are underlined):

$$\left\{\begin{array}{lllllll}
\underline{a_1b_1a_1^{-1}b_1^{-1}}, & \underline{a_1b_2a_1^{-1}b_3^{-1}}, & \underline{a_1b_3a_2b_2^{-1}}, & a_1b_4a_7^{-1}b_4^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_6a_5^{-1}b_6^{-1}, & a_1b_7a_1^{-1}b_7^{-1}, \\
a_1b_6^{-1}a_5^{-1}b_6, & a_1b_4^{-1}a_7^{-1}b_4, & \underline{a_1b_3^{-1}a_3^{-1}b_2}, & \underline{a_2b_1a_3^{-1}b_2^{-1}}, & \underline{a_2b_2a_3^{-1}b_3^{-1}}, & \underline{a_2b_3a_3^{-1}b_1}, & a_2b_4a_2^{-1}b_4^{-1}, \\
a_2b_5a_2^{-1}b_5^{-1}, & a_2b_6a_2^{-1}b_6^{-1}, & a_2b_7a_2^{-1}b_7^{-1}, & \underline{a_2b_3^{-1}a_3b_2}, & \underline{a_2b_1^{-1}a_3^{-1}b_1^{-1}}, & a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3^{-1}b_5^{-1}, \\
a_3b_6a_3^{-1}b_6^{-1}, & a_3b_7a_3^{-1}b_7^{-1}, & a_4b_1a_4^{-1}b_7^{-1}, & a_4b_2a_4^{-1}b_2^{-1}, & a_4b_3a_4^{-1}b_3^{-1}, & a_4b_4a_5^{-1}b_5^{-1}, & a_4b_5a_5^{-1}b_4^{-1}, \\
a_4b_6a_4^{-1}b_6^{-1}, & a_4b_7a_4^{-1}b_1^{-1}, & a_4b_5^{-1}a_5^{-1}b_4, & a_4b_4^{-1}a_5^{-1}b_5, & a_5b_1a_5^{-1}b_1^{-1}, & a_5b_2a_5^{-1}b_2^{-1}, & a_5b_3a_5^{-1}b_3^{-1}, \\
a_5b_7a_5^{-1}b_7^{-1}, & a_6b_1a_6^{-1}b_5^{-1}, & a_6b_2a_6^{-1}b_2^{-1}, & a_6b_3a_6^{-1}b_3^{-1}, & a_6b_4a_6^{-1}b_4^{-1}, & a_6b_5a_6^{-1}b_1^{-1}, & a_6b_6a_7^{-1}b_7^{-1}, \\
a_6b_7a_7^{-1}b_6^{-1}, & a_6b_7^{-1}a_7^{-1}b_6, & a_6b_6^{-1}a_7^{-1}b_7, & a_7b_1a_7^{-1}b_1^{-1}, & a_7b_2a_7^{-1}b_2^{-1}, & a_7b_3a_7^{-1}b_3^{-1}, & a_7b_5a_7^{-1}b_5^{-1}
\end{array}\right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (2,3)(12,13), \\
\rho_v(b_2) &= (1,13,12,2,3), \\
\rho_v(b_3) &= (2,3,13,12,14), \\
\rho_v(b_4) &= (1,7)(4,5)(8,14)(10,11), \\
\rho_v(b_5) &= (4,5)(10,11), \\
\rho_v(b_6) &= (1,5)(6,7)(8,9)(10,14), \\
\rho_v(b_7) &= (6,7)(8,9),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (2,3)(12,13), \\
\rho_h(a_2) &= (1,14,3,2)(12,13), \\
\rho_h(a_3) &= (1,12,13,14)(2,3), \\
\rho_h(a_4) &= (1,7)(4,5)(8,14)(10,11), \\
\rho_h(a_5) &= (4,5)(10,11), \\
\rho_h(a_6) &= (1,5)(6,7)(8,9)(10,14), \\
\rho_h(a_7) &= (6,7)(8,9).
\end{aligned}$$

A.4 Proof of Proposition 16

We first recall Proposition 16:

Proposition 16. *For each $k \in \mathbb{N}_0$ and Γ as in Example 1 we have $\langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma = \Gamma_0$.*

In the proof we need two lemmas:

Lemma 76. $b_3^{-1} b_2 a_1^{6(1+2k)} b_2^{-1} b_3 = a_2^{-6(1+2k)}$ *for each $k \in \mathbb{N}_0$.*

Proof. We only use the given relators in Γ. For $k = 0$:

$$\begin{aligned}
& b_3^{-1} b_2 a_1^6 b_2^{-1} b_3 \\
&= b_3^{-1} a_3 b_3 a_1^5 b_2^{-1} b_3 \\
&= b_3^{-1} a_3 a_1 b_2 a_1^4 b_2^{-1} b_3 \\
&= b_3^{-1} a_3 a_1 a_3 b_3 a_1^3 b_2^{-1} b_3 \\
&= b_3^{-1} a_3 a_1 a_3 a_1 b_2 a_1^2 b_2^{-1} b_3 \\
&= b_3^{-1} a_3 a_1 a_3 a_1 a_3 b_3 a_1 b_2^{-1} b_3 \\
&= b_3^{-1} a_3 a_1 a_3 a_1 a_3 a_1 b_2 b_2^{-1} b_3 \\
&= a_2^{-1} b_2^{-1} a_1 a_3 a_1 a_3 a_1 b_3 \\
&= a_2^{-1} a_2^{-1} b_3^{-1} a_3 a_1 a_3 a_1 b_3 \\
&= a_2^{-2} a_2^{-1} b_2^{-1} a_1 a_3 a_1 b_3 \\
&= a_2^{-3} a_2^{-1} b_3^{-1} a_3 a_1 b_3 \\
&= a_2^{-4} a_2^{-1} b_2^{-1} a_1 b_3 \\
&= a_2^{-5} a_2^{-1} b_3^{-1} b_3 \\
&= a_2^{-6}.
\end{aligned}$$

Therefore $b_3^{-1} b_2 a_1^{6(1+2k)} b_2^{-1} b_3 = a_2^{-6(1+2k)}$. ☐

Lemma 77. $a_2 b_3 b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} = a_2^{6(1+2k)} b_2 b_1$, *$k \in \mathbb{N}_0$.*

Proof. The proof is by induction on k.
$k = 0$:

$$\begin{aligned}
& a_2 b_3 b_2 b_3^{-1} a_1^6 b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 a_1 b_2^{-1} a_1^5 b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 a_1 a_2^{-1} b_3^{-1} a_1^4 b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 a_1 a_2^{-1} a_1 b_2^{-1} a_1^3 b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 a_1 a_2^{-1} a_1 a_2^{-1} b_3^{-1} a_1^2 b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 a_1 a_2^{-1} a_1 a_2^{-1} a_1 b_2^{-1} a_1 b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 a_1 a_2^{-1} a_1 a_2^{-1} a_1 a_2^{-1} b_3^{-1} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 a_3 b_3 a_2^{-1} a_1 a_2^{-1} a_1 a_2^{-1} b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 a_3 a_3 b_2 a_1 a_2^{-1} a_1 a_2^{-1} b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 a_3^2 a_3 b_3 a_2^{-1} a_1 a_2^{-1} b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 a_3^3 a_3 b_2 a_1 a_2^{-1} b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 a_3^4 a_3 b_3 a_2^{-1} b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 a_3^5 a_3 b_2 b_2^{-1} b_3^{-1} a_2^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= a_2a_2b_2a_3^5b_3^{-1}a_2^{-1}\\
&= a_2^2a_2b_1a_3^4b_3^{-1}a_2^{-1}\\
&= a_2^3a_2b_1^{-1}a_3^3b_3^{-1}a_2^{-1}\\
&= a_2^4a_2b_3a_3^2b_3^{-1}a_2^{-1}\\
&= a_2^5a_2b_2a_3b_3^{-1}a_2^{-1}\\
&= a_2^6a_2b_1b_3^{-1}a_2^{-1}\\
&= a_2^7b_1a_3^{-1}b_1\\
&= a_2^7a_2^{-1}b_2b_1\\
&= a_2^6b_2b_1.
\end{aligned}
$$

induction step $k \to k+1$:

$$
\begin{aligned}
&a_2b_3b_2b_3^{-1}a_1^{6(1+2(k+1))}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2b_3^{-1}a_1^{12}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2a_1b_2^{-1}a_1^{11}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2a_1a_2^{-1}b_3^{-1}a_1^{10}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2a_1a_2^{-1}a_1b_2^{-1}a_1^{9}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2a_1a_2^{-1}a_1a_2^{-1}b_3^{-1}a_1^{8}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^2a_1b_2^{-1}a_1^{7}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^2a_1a_2^{-1}b_3^{-1}a_1^{6}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^3a_1b_2^{-1}a_1^{5}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^3a_1a_2^{-1}b_3^{-1}a_1^{4}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^4a_1b_2^{-1}a_1^{3}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^4a_1a_2^{-1}b_3^{-1}a_1^{2}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^5a_1b_2^{-1}a_1a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3b_2(a_1a_2^{-1})^5a_1a_2^{-1}b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3b_3a_2^{-1}(a_1a_2^{-1})^5b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3a_3b_2(a_1a_2^{-1})^5b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^2a_3b_3a_2^{-1}(a_1a_2^{-1})^4b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^3a_3b_2(a_1a_2^{-1})^4b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^4a_3b_3a_2^{-1}(a_1a_2^{-1})^3b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^5a_3b_2(a_1a_2^{-1})^3b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^6a_3b_3a_2^{-1}(a_1a_2^{-1})^2b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^7a_3b_2(a_1a_2^{-1})^2b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^8a_3b_3a_2^{-1}a_1a_2^{-1}b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^9a_3b_2a_1a_2^{-1}b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^{10}a_3b_3a_2^{-1}b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}\\
&= a_2b_3a_3^{11}a_3b_2b_3^{-1}a_1^{6(1+2k)}b_3b_2^{-1}b_3^{-1}a_2^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= a_2 a_2 b_2 a_3^{11} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^2 a_2 b_1 a_3^{10} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^3 a_2 b_1^{-1} a_3^{9} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^4 a_2 b_3 a_3^{8} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^5 a_2 b_2 a_3^{7} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^6 a_2 b_1 a_3^{6} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^7 a_2 b_1^{-1} a_3^{5} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^8 a_2 b_3 a_3^{4} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^9 a_2 b_2 a_3^{3} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^{10} a_2 b_1 a_3^{2} b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^{11} a_2 b_1^{-1} a_3 b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^{12} a_2 b_3 b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^{12} a_2^{6(1+2k)} b_2 b_1 \quad \text{(by the induction hypothesis)} \\
&= a_2^{6(1+2(k+1))} b_2 b_1.
\end{aligned}
$$

□

Proof of Proposition 16. Since $a_1^2 \in \Gamma_0$, one inclusion is obvious:

$$\langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma < \Gamma_0.$$

For the other inclusion we have by Lemma 76

$$a_2^{-6(1+2k)} \in \langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma,$$

and by Lemma 77

$$a_2^{6(1+2k)} b_2 b_1 \in \langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma,$$

hence together

$$b_2 b_1 \in \langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma. \tag{20}$$

Next, we observe that $b_1^2 \in \langle\langle b_2 b_1 \rangle\rangle_\Gamma$ since

$$
\begin{aligned}
&(a_1 a_2^{-1} b_2 b_1 a_2 a_1^{-1})(a_1^2 b_2 b_1 a_1^{-2}) \\
&= a_1 a_2^{-1} b_2 b_1 a_2 a_1 b_2 b_1 a_1^{-2} \\
&= a_1 a_2^{-1} b_2 b_1 a_2 a_1 b_2 a_1^{-2} b_1 \\
&= a_1 a_2^{-1} b_2 b_1 a_2 a_1 a_1^{-1} b_3 a_1^{-1} b_1 \\
&= a_1 a_2^{-1} b_2 b_1 a_2 a_1^{-1} b_0 b_1 \\
&= a_1 a_2^{-1} b_2 a_3 b_3^{-1} a_3^{-1} b_2 b_1 \\
&= a_1 a_2^{-1} b_2 a_3 a_1^{-1} b_2^{-1} b_2 b_1 \\
&= a_1 a_2^{-1} b_2 a_3 a_1^{-1} b_1 \\
&= a_1 a_2^{-1} a_2 b_1 a_1^{-1} b_1 \\
&= a_1 b_1 a_1^{-1} b_1 \\
&= a_1 a_1^{-1} b_1 b_1 \\
&= b_1^2.
\end{aligned}
$$

Moreover, $a_1a_3^{-1} \in \langle\langle b_1^2 \rangle\rangle_\Gamma < \langle\langle b_2b_1 \rangle\rangle_\Gamma$, since

$$\begin{aligned}
&(a_1a_2^{-1}b_1^{-2}a_2a_1^{-1})(a_1^{-1}a_2^{-1}b_1^2a_2a_1)\\
&= a_1a_2^{-1}b_1^{-2}a_2a_1^{-2}a_2^{-1}b_1^2a_2a_1\\
&= a_1a_2^{-1}b_1^{-2}a_2a_1^{-2}a_2^{-1}b_1a_3b_3^{-1}a_1\\
&= a_1a_2^{-1}b_1^{-2}a_2a_1^{-2}a_2^{-1}b_1a_3a_1b_2^{-1}\\
&= a_1a_2^{-1}b_1^{-2}a_2a_1^{-2}a_2^{-1}a_2b_1^{-1}a_1b_2^{-1}\\
&= a_1a_2^{-1}b_1^{-2}a_2a_1^{-2}a_1b_1^{-1}b_2^{-1}\\
&= a_1a_2^{-1}b_1^{-1}a_3b_1a_1^{-1}b_1^{-1}b_2^{-1}\\
&= a_1a_2^{-1}b_1^{-1}a_3a_1^{-1}b_1b_1^{-1}b_2^{-1}\\
&= a_1a_2^{-1}a_2b_3a_1^{-1}b_2^{-1}\\
&= a_1a_3^{-1}b_2b_2^{-1}\\
&= a_1a_3^{-1}.
\end{aligned}$$

It is easy to check that Γ_0 is generated (as a subgroup of Γ) by $\{a_1a_3^{-1}, b_1^2\}$. We conclude that

$$\Gamma_0 = \langle a_1a_3^{-1}, b_1^2 \rangle < \langle\langle b_2b_1 \rangle\rangle_\Gamma \overset{(20)}{<} \langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma,$$

as required. $\square$

A.5 Amalgam decomposition of Example 3

We give a finite presentation of the horizontal decomposition $\Gamma_0 \cong F_5 *_{F_{41}} F_5$ in Example 3:

$$\text{generators}(\Gamma_0) = \{s_1, s_2, s_3, s_4, s_5, u_1, u_2, u_3, u_4, u_5\},$$

$$\text{relations}(\Gamma_0) = \left\{ \begin{array}{rcl}
s_1^{-1}s_3s_4^{-1}s_3 & = & u_4^{-1}u_1u_4u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_4s_1^{-1} & = & u_3^{-1}u_1u_3u_1u_3u_1, \\
s_3^{-1}s_4^2s_3^3s_4^{-2}s_3 & = & u_3^{-1}u_1u_3u_4^{-1}u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3s_1s_3s_4^{-2}s_3 & = & u_4^{-1}u_1^{-1}u_2u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_1s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-1}u_2u_3, \\
s_3s_2s_3s_4^{-2}s_3 & = & u_4^{-1}u_1^{-1}u_5^{-1}u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_2s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_1^{-1}u_2u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4s_2s_4s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_5^{-1}u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_1^{-1}s_2s_3s_4^{-1}s_3 & = & u_4^{-1}u_2u_4u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_3^{-1}s_4^{-1}s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_4^{-1}u_1^{-1}u_3, \\
s_1^{-1}s_3^{-1}s_4s_3^2s_4^{-2}s_3 & = & u_4^{-2}u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_4s_3^{-1}s_1^{-1} & = & u_3^{-1}u_1u_3u_1u_3u_4^{-1}, \\
s_4^{-1}s_3^3s_4^{-2}s_3 & = & u_5^{-1}u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_4^{-1}s_3^2 & = & u_3^{-1}u_1u_3u_1u_5^{-1}u_4u_1u_3, \\
s_5^{-1}s_3^3s_4^{-2}s_3 & = & u_2u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_5^{-1}s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_1^{-1}u_5^{-1}u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4s_5^{-1}s_4s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_2u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_5^{-1}s_4^{-1}s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_5^{-1}u_1^{-1}u_3, \\
s_3s_4^{-1}s_1^{-1} & = & u_4^{-1}u_1^{-1}u_3^{-1}, \\
s_1^{-1}s_4^{-1}s_3s_4^{-1}s_3 & = & u_4^{-1}u_3^{-1}u_4u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_4^{-1} & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_3^{-1}u_4, \\
s_3^{-1}s_4s_1s_3^2 & = & u_3^{-1}u_3^{-1}u_4u_1u_3, \\
s_3^{-1}s_4^{-1}s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_1^{-1}u_3^{-1}u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_1s_4s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_1u_3u_1u_3^{-1}u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_5^{-1}s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-1}u_4^{-1}u_3, \\
s_3s_5^{-1}s_3s_4^{-2}s_3 & = & u_4^{-1}u_1^{-1}u_4^{-1}u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_2s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-1}u_5^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_4s_5^{-1}s_1^{-1} & = & u_3^{-1}u_1u_3u_1u_3u_5^{-1}, \\
s_3^{-1}s_4^2s_2 & = & u_3^{-1}u_1u_3u_5^{-1}u_4, \\
s_1^{-1}s_5^{-1}s_3s_4^{-1}s_3 & = & u_4^{-1}u_5^{-1}u_4u_3, \\
s_3^3 & = & u_4^{-1}u_1^{-1}u_4u_1u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_4s_1s_3^3s_4^{-2}s_3 & = & u_3^{-1}u_1u_3u_1u_3^2u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_1s_4^{-1}s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_1s_3s_4^{-1}s_3 & = & u_3^{-1}u_1u_3u_4u_3, \\
s_3^{-1}s_1s_3^2s_4^{-2}s_3 & = & u_3^{-1}u_1^{-1}u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_2s_3^2 & = & u_3^{-1}u_1u_3u_1^2u_4u_1u_3, \\
s_2s_3^3s_4^{-2}s_3 & = & u_1u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_5^{-1}s_3^2 & = & u_3^{-1}u_1u_3u_1u_2u_4u_1u_3, \\
s_3^{-1}s_4^2s_3^{-1}s_4s_2s_1^{-1} & = & u_3^{-1}u_1u_3u_1u_3u_2, \\
s_3^{-1}s_4^2s_5^{-1} & = & u_3^{-1}u_1u_3u_2u_4, \\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_2s_4^{-1}s_3 & = & u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_2u_1^{-1}u_3
\end{array} \right\}.$$

A.6 Table: $\left|\rho_v^{(k)}(w)\right|$ in Example 3 for $|w| = 2,\ k \leq 5$

Note that

$$|\rho_v^{(k)}(b\tilde{b})| = |\rho_v^{(k)}(\tilde{b}b)| = |\rho_v^{(k)}(b\tilde{b})^{-1}| = |\rho_v^{(k)}(\tilde{b}b)^{-1}|$$

if $b, \tilde{b} \in \{b_1, \ldots, b_5, b_5^{-1}, \ldots, b_1^{-1}\}$.

$\left|\rho_v^{(k)}(w)\right|$	$k = 1$	2	3	4	5
$w = b_1^2$	5	**5**	50	300	1500
b_1b_2	3	15	75	150	2250
b_1b_3	5	10	150	900	9000
b_1b_4	3	15	30	450	4500
b_1b_5	5	30	300	900	5400
$b_1b_5^{-1}$	5	15	450	4500	**4500**
$b_1b_4^{-1}$	5	15	150	900	1800
$b_1b_3^{-1}$	5	25	50	500	3000
$b_1b_2^{-1}$	3	9	54	**54**	1620
b_2^2	5	**5**	50	300	1500
b_2b_3	5	25	50	500	3000
b_2b_4	5	15	150	900	1800
b_2b_5	5	30	300	900	5400
$b_2b_5^{-1}$	5	15	450	4500	**4500**
$b_2b_4^{-1}$	3	15	30	450	4500
$b_2b_3^{-1}$	5	10	150	900	9000
b_3^2	1	5	25	50	500
b_3b_4	2	6	90	180	2700
b_3b_5	1	30	**30**	450	4500
$b_3b_5^{-1}$	1	30	**30**	450	4500
$b_3b_4^{-1}$	2	20	60	600	1800
b_4^2	2	4	20	100	500
b_4b_5	2	10	20	600	6000
$b_4b_5^{-1}$	2	10	20	600	6000
b_5^2	1	2	10	20	600

B Supplement to Section 5

B.1 Amalgam decompositions of Example 50

We first give the vertical decomposition of Γ of Example 50:

$$\Gamma \cong F_3^{(b)} *_{F_{17}^{(b)} \cong F_{17}^{(s)}} (\mathbb{Z}_2^{*12} * F_3^{(s)}).$$

$$F_3^{(b)} = \langle b_1, b_2, b_3 \rangle,$$
$$\mathbb{Z}_2^{*12} * F_3^{(s)} = \langle s_1, \ldots, s_{12}, s_{13}, s_{14}, s_{15} \mid s_1^2 = \ldots = s_{12}^2 = 1 \rangle.$$

The subgroup $F_{17}^{(b)} < F_3^{(b)}$ of index 8 is given by

$$\begin{aligned} F_{17}^{(b)} = \langle & b_1^{-1}b_2,\, b_1^{-1}b_3,\, b_2b_1b_3^{-1},\, b_1^2b_2b_1,\, b_1b_2^2b_1, \\ & b_1b_3^{-1}b_2b_1,\, b_1^{-1}b_2^{-1}b_1b_2b_1^2,\, b_1^{-1}b_2^{-1}b_3^{-1}b_1^2,\, b_3b_1^3, \\ & b_3^2b_1^2,\, b_3b_2^{-1}b_1^2,\, b_3b_1^{-1}b_2^2b_1^2,\, b_3b_1^{-1}b_3b_2b_1^2, \\ & b_3b_1^{-2}b_2b_1^2,\, b_1^{-1}b_3^{-1}b_2b_1^2,\, b_1b_2^{-1}b_3^{-1},\, b_1b_3b_1b_3^{-1} \rangle, \end{aligned}$$

the index 2 subgroup $F_{17}^{(s)} < \mathbb{Z}_2^{*12} * F_3^{(s)}$ by

$$\begin{aligned} F_{17}^{(s)} = \langle & s_1s_2,\, s_1s_3,\, s_{13},\, s_4s_1,\, s_5s_1,\, s_6s_1, \\ & s_1s_{14}s_1,\, s_1s_{15}s_1,\, s_7s_1,\, s_8s_1,\, s_9s_1, \\ & s_{10}s_1,\, s_{11}s_1,\, s_{12}s_1,\, s_1s_{13}s_1,\, s_{15},\, s_{14} \rangle. \end{aligned}$$

The identification in Γ is

$$\begin{array}{rcl} F_{17}^{(b)} & \overset{\cong}{\longleftrightarrow} & F_{17}^{(s)} \\ \\ b_1^{-1}b_2 & \longleftrightarrow & s_1s_2 \\ b_1^{-1}b_3 & \longleftrightarrow & s_1s_3 \\ b_2b_1b_3^{-1} & \longleftrightarrow & s_{13} \\ b_1^2b_2b_1 & \longleftrightarrow & s_4s_1 \\ b_1b_2^2b_1 & \longleftrightarrow & s_5s_1 \\ b_1b_3^{-1}b_2b_1 & \longleftrightarrow & s_6s_1 \\ b_1^{-1}b_2^{-1}b_1b_2b_1^2 & \longleftrightarrow & s_1s_{14}s_1 \\ b_1^{-1}b_2^{-1}b_3^{-1}b_1^2 & \longleftrightarrow & s_1s_{15}s_1 \\ b_3b_1^3 & \longleftrightarrow & s_7s_1 \\ b_3^2b_1^2 & \longleftrightarrow & s_8s_1 \\ b_3b_2^{-1}b_1^2 & \longleftrightarrow & s_9s_1 \\ b_3b_1^{-1}b_2^2b_1^2 & \longleftrightarrow & s_{10}s_1 \\ b_3b_1^{-1}b_3b_2b_1^2 & \longleftrightarrow & s_{11}s_1 \\ b_3b_1^{-2}b_2b_1^2 & \longleftrightarrow & s_{12}s_1 \\ b_1^{-1}b_3^{-1}b_2b_1^2 & \longleftrightarrow & s_1s_{13}s_1 \\ b_1b_2^{-1}b_3^{-1} & \longleftrightarrow & s_{15} \\ b_1b_3b_1b_3^{-1} & \longleftrightarrow & s_{14} \end{array}$$

Recall the presentation of Γ given in Section 5.4.2: $\Gamma = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid R \rangle$, where

$$R = \left\{ \begin{array}{lll} \mathbf{a_1 b_1 a_1 b_1}, & \mathbf{a_1 b_2 a_1 b_2}, & \mathbf{a_1 b_3 a_1 b_3}, \\ a_1 b_3^{-1} a_4 b_2^{-1}, & a_1 b_2^{-1} a_2 b_1^{-1}, & a_1 b_1^{-1} a_3 b_3^{-1}, \\ \mathbf{a_2 b_1 a_2 b_1}, & \mathbf{a_2 b_2 a_2 b_2}, & a_2 b_3 a_4^{-1} b_1^{-1}, \\ \mathbf{a_2 b_3^{-1} a_2 b_3^{-1}}, & a_2 b_2^{-1} a_3^{-1} b_3, & \mathbf{a_3 b_1 a_3 b_1}, \\ \mathbf{a_3 b_3 a_3 b_3}, & \mathbf{a_3 b_2^{-1} a_3 b_2^{-1}}, & a_3 b_1^{-1} a_4^{-1} b_2, \\ \mathbf{a_4 b_2 a_4 b_2}, & \mathbf{a_4 b_3 a_4 b_3}, & \mathbf{a_4 b_1^{-1} a_4 b_1^{-1}} \end{array} \right\}.$$

The isomorphism to the amalgam described above is

$$\begin{array}{ccc} F_3^{(b)} *_{F_{17}^{(b)} \cong F_{17}^{(s)}} (\mathbb{Z}_2^{*12} * F_3^{(s)}) & \overset{\cong}{\longleftrightarrow} & \Gamma = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid R \rangle \\ \\ s_1 & \longleftrightarrow & a_1 b_1 \\ s_2 & \longleftrightarrow & a_1 b_2 \\ s_3 & \longleftrightarrow & a_1 b_3 \\ s_4 & \longleftrightarrow & a_1 b_2^{-1} b_1^{-2} \\ s_5 & \longleftrightarrow & a_1 b_2^{-2} b_1^{-1} \\ s_6 & \longleftrightarrow & a_1 b_2^{-1} b_3 b_1^{-1} \\ s_7 & \longleftrightarrow & a_1 b_1^{-2} b_3^{-1} \\ s_8 & \longleftrightarrow & a_1 b_1^{-1} b_3^{-2} \\ s_9 & \longleftrightarrow & a_1 b_1^{-1} b_2 b_3^{-1} \\ s_{10} & \longleftrightarrow & a_1 b_1^{-1} b_2^{-2} b_1 b_3^{-1} \\ s_{11} & \longleftrightarrow & a_1 b_1^{-1} b_2^{-1} b_3^{-1} b_1 b_3^{-1} \\ s_{12} & \longleftrightarrow & a_1 b_1^{-1} b_2^{-1} b_1^2 b_3^{-1} \\ s_{13} & \longleftrightarrow & b_2 b_1 b_3^{-1} \\ s_{14} & \longleftrightarrow & b_1 b_3 b_1 b_3^{-1} \\ s_{15} & \longleftrightarrow & b_1 b_2^{-1} b_3^{-1} \\ s_1 b_1^{-1} & \longleftrightarrow & a_1 \\ b_1^{-2} s_4 b_1 & \longleftrightarrow & a_2 \\ b_3^{-2} s_8 b_3 & \longleftrightarrow & a_3 \\ b_2^{-1} b_1 b_3^{-1} s_{10} b_3 b_1^{-1} & \longleftrightarrow & a_4 \\ b_1 & \longleftrightarrow & b_1 \\ b_2 & \longleftrightarrow & b_2 \\ b_3 & \longleftrightarrow & b_3 \end{array}$$

We decribe now the (vertical) amalgam decomposition

$$\Gamma_0 \cong F_5^{(r)} *_{F_{33}^{(r)} \cong F_{33}^{(q)}} F_5^{(q)}.$$

$$\begin{aligned} F_5^{(r)} &= \langle r_1, r_2, r_3, r_4, r_5 \rangle, \\ F_5^{(q)} &= \langle q_1, q_2, q_3, q_4, q_5 \rangle, \end{aligned}$$

$$
\begin{aligned}
F_{33}^{(r)} = \langle & r_3^{-1}r_5,\, r_4^{-1}r_5,\, r_5r_1r_5,\, r_4r_1r_5,\, r_2^{-1}r_1r_5,\, r_1r_4r_2r_5,\, r_1r_3r_2r_5,\, r_1r_2r_5, \\
& r_2r_5^2,\, r_2r_3r_5,\, r_2r_1^{-1}r_5,\, r_5^{-1}r_1^{-2}r_3^{-1},\, r_5^{-1}r_1^{-1}r_2^{-1}r_3^{-1}, r_5^{-1}r_1^{-1}r_5r_3^{-1},\, r_1^{-1}r_3r_1r_5, \\
& r_1^{-1}r_2r_3r_1r_5,\, r_1^{-1}r_4^{-1}r_3r_1r_5,\, r_2r_4r_5r_2r_5,\, r_2r_4r_1r_5r_2r_5,\, r_2r_4r_3^{-1}r_5r_2r_5, \\
& r_1^{-1}r_3^{-2}r_1r_5,\, r_1^{-1}r_3^{-1}r_2^{-1}r_3^{-1}r_1r_5,\, r_1^{-1}r_3^{-1}r_1^{-1}r_3^{-1}r_1r_5,\, r_2r_4r_2r_1^{-1}, \\
& r_1^{-1}r_3^{-1}r_5r_4^{-1}r_2^{-1},\, r_5^{-1}r_1^{-1}r_3r_5^{-1}r_3r_1r_5,\, r_1^{-1}r_3^{-1}r_4r_3^{-1},\, r_1^{-1}r_5^{-1}r_1^{-1}, \\
& r_5^{-1}r_2^{-1}r_1r_3r_1r_5,\, r_5^{-1}r_1^{-1}r_4r_5,\, r_5^{-1}r_2r_5r_2r_5,\, r_3r_2^{-1},\, r_5^{-1}r_1^{-1}r_3r_4^{-1}r_5r_2r_5\rangle,
\end{aligned}
$$

$$
\begin{aligned}
F_{33}^{(q)} = \langle & q_2,\, q_1,\, q_4^{-1}q_5^{-1},\, q_4^{-1}q_1^{-1}q_5^{-1},\, q_4^{-1}q_3q_5^{-1},\, q_3^{-1}q_1^{-1}q_4^{-1},\, q_3^{-1}q_2^{-1}q_4^{-1}, \\
& q_3^{-1}q_5q_4^{-1},\, q_5^{-1}q_3^{-1},\, q_5^{-1}q_2^{-1}q_3^{-1},\, q_5^{-1}q_4q_3^{-1},\, q_5q_2q_4q_5^{-1}q_4,\, q_5q_2q_3q_5^{-1}q_4, \\
& q_5q_2q_5^{-1}q_4,\, q_4^{-1}q_2^{-1}q_5^{-1}q_4q_5^{-1},\, q_4^{-1}q_2^{-1}q_3^{-1}q_4q_5^{-1},\, q_4^{-1}q_2^{-1}q_1q_4q_5^{-1}, \\
& q_5^{-1}q_3q_5^{-1}q_3q_4^{-1},\, q_5^{-1}q_3q_4^{-1}q_3q_4^{-1},\, q_5^{-1}q_3q_2q_3q_4^{-1},\, q_4^{-1}q_2^{-1}q_5q_1q_2^{-1}q_5^{-1}, \\
& q_4^{-1}q_2^{-1}q_3q_1q_2^{-1}q_5^{-1},\, q_4^{-1}q_2^{-1}q_4q_1q_2^{-1}q_5^{-1},\, q_5^{-1}q_3^2,\, q_4^{-1}q_2^{-1}q_1^{-1}q_3^{-1}q_5, \\
& q_5q_2q_1^{-1}q_2q_4q_5^{-1},\, q_4^{-1}q_2^{-2}q_5^{-1}q_4,\, q_4^{-1}q_2^{-1}q_4^{-1}q_3,\, q_4^2q_5^{-1},\, q_5q_2q_5q_3^{-1}, \\
& q_3q_1q_3q_4^{-1},\, q_4^{-1}q_5q_1q_5,\, q_5q_2q_1^{-1}q_3q_4^{-1}\rangle,
\end{aligned}
$$

$$
\begin{array}{rcl}
F_{33}^{(r)} & \xleftrightarrow{\cong} & F_{33}^{(q)} \\
\\
r_3^{-1}r_5 & \longleftrightarrow & q_2 \\
r_4^{-1}r_5 & \longleftrightarrow & q_1 \\
r_5r_1r_5 & \longleftrightarrow & q_4^{-1}q_5^{-1} \\
r_4r_1r_5 & \longleftrightarrow & q_4^{-1}q_1^{-1}q_5^{-1} \\
r_2^{-1}r_1r_5 & \longleftrightarrow & q_4^{-1}q_3q_5^{-1} \\
r_1r_4r_2r_5 & \longleftrightarrow & q_3^{-1}q_1^{-1}q_4^{-1} \\
r_1r_3r_2r_5 & \longleftrightarrow & q_3^{-1}q_2^{-1}q_4^{-1} \\
r_1r_2r_5 & \longleftrightarrow & q_3^{-1}q_5q_4^{-1} \\
r_2r_5^2 & \longleftrightarrow & q_5^{-1}q_3^{-1} \\
r_2r_3r_5 & \longleftrightarrow & q_5^{-1}q_2^{-1}q_3^{-1} \\
r_2r_1^{-1}r_5 & \longleftrightarrow & q_5^{-1}q_4q_3^{-1} \\
r_5^{-1}r_1^{-2}r_3^{-1} & \longleftrightarrow & q_5q_2q_4q_5^{-1}q_4 \\
r_5^{-1}r_1^{-1}r_2^{-1}r_3^{-1} & \longleftrightarrow & q_5q_2q_3q_5^{-1}q_4 \\
r_5^{-1}r_1^{-1}r_5r_3^{-1} & \longleftrightarrow & q_5q_2q_5^{-1}q_4 \\
r_1^{-1}r_3r_1r_5 & \longleftrightarrow & q_4^{-1}q_2^{-1}q_5^{-1}q_4q_5^{-1} \\
r_1^{-1}r_2r_3r_1r_5 & \longleftrightarrow & q_4^{-1}q_2^{-1}q_3^{-1}q_4q_5^{-1} \\
r_1^{-1}r_4^{-1}r_3r_1r_5 & \longleftrightarrow & q_4^{-1}q_2^{-1}q_1q_4q_5^{-1} \\
r_2r_4r_5r_2r_5 & \longleftrightarrow & q_5^{-1}q_3q_5^{-1}q_3q_4^{-1} \\
r_2r_4r_1r_5r_2r_5 & \longleftrightarrow & q_5^{-1}q_3q_4^{-1}q_3q_4^{-1} \\
r_2r_4r_3^{-1}r_5r_2r_5 & \longleftrightarrow & q_5^{-1}q_3q_2q_3q_4^{-1} \\
r_1^{-1}r_3^{-2}r_1r_5 & \longleftrightarrow & q_4^{-1}q_2^{-1}q_5q_1q_2^{-1}q_5^{-1} \\
r_1^{-1}r_3^{-1}r_2^{-1}r_3^{-1}r_1r_5 & \longleftrightarrow & q_4^{-1}q_2^{-1}q_3q_1q_2^{-1}q_5^{-1} \\
r_1^{-1}r_3^{-1}r_1^{-1}r_3^{-1}r_1r_5 & \longleftrightarrow & q_4^{-1}q_2^{-1}q_4q_1q_2^{-1}q_5^{-1} \\
r_2r_4r_2r_1^{-1} & \longleftrightarrow & q_5^{-1}q_3^2 \\
r_1^{-1}r_3^{-1}r_5r_4^{-1}r_2^{-1} & \longleftrightarrow & q_4^{-1}q_2^{-1}q_1^{-1}q_3^{-1}q_5 \\
r_5^{-1}r_1^{-1}r_3r_5^{-1}r_3r_1r_5 & \longleftrightarrow & q_5q_2q_1^{-1}q_2q_4q_5^{-1} \\
r_1^{-1}r_3^{-1}r_4r_3^{-1} & \longleftrightarrow & q_4^{-1}q_2^{-2}q_5^{-1}q_4 \\
r_1^{-1}r_5^{-1}r_1^{-1} & \longleftrightarrow & q_4^{-1}q_2^{-1}q_4^{-1}q_3 \\
r_5^{-1}r_2^{-1}r_1r_3r_1r_5 & \longleftrightarrow & q_4^2q_5^{-1} \\
r_5^{-1}r_1^{-1}r_4r_5 & \longleftrightarrow & q_5q_2q_5q_3^{-1} \\
r_5^{-1}r_2r_5r_2r_5 & \longleftrightarrow & q_3q_1q_3q_4^{-1} \\
r_3r_2^{-1} & \longleftrightarrow & q_4^{-1}q_5q_1q_5 \\
r_5^{-1}r_1^{-1}r_3r_4^{-1}r_5r_2r_5 & \longleftrightarrow & q_5q_2q_1^{-1}q_3q_4^{-1}.
\end{array}
$$

The isomorphism is

$$
\begin{array}{ccc}
F_5^{(r)} *_{F_{33}^{(r)} \cong F_{33}^{(q)}} F_5^{(q)} & \xleftrightarrow{\cong} & \Gamma_0 < \Gamma \\
\\
r_1 & \longleftrightarrow & b_2b_1^{-1} \\
r_2 & \longleftrightarrow & b_3b_1^{-1} \\
r_3 & \longleftrightarrow & b_1b_3 \\
r_4 & \longleftrightarrow & b_1b_2 \\
r_5 & \longleftrightarrow & b_1^2 \\
q_1 & \longleftrightarrow & b_2^{-1}b_1 \\
q_2 & \longleftrightarrow & b_3^{-1}b_1 \\
q_3 & \longleftrightarrow & a_1a_2b_3b_2^{-1} \\
q_4 & \longleftrightarrow & a_1a_2^{-1}b_1^{-2} \\
q_5 & \longleftrightarrow & a_1a_3^{-1}b_1^{-1}b_3^{-1}
\end{array}
$$

B.2 Amalgam decompositions of Example 52

$$F_3^{(v,b)} *_{F_9^{(v,b)} \cong F_9^{(v,s)}} F_5^{(v,s)} \overset{\cong}{\longleftrightarrow} \langle a_1, \ldots, b_3 \mid R(2,3) \rangle \overset{\cong}{\longleftrightarrow} F_2^{(h,a)} *_{F_7^{(h,a)} \cong F_7^{(h,u)}} F_4^{(h,u)}$$

$$\begin{array}{rcccl}
s_4 b_3 & \longleftrightarrow & a_1 & \longleftrightarrow & a_1 \\
b_1 s_2 b_2^{-1} & \longleftrightarrow & a_2 & \longleftrightarrow & a_2 \\
b_1 & \longleftrightarrow & b_1 & \longleftrightarrow & u_2^{-1} a_2 a_1 \\
b_2 & \longleftrightarrow & b_2 & \longleftrightarrow & a_2^2 u_2^{-1} a_2^2 \\
b_3 & \longleftrightarrow & b_3 & \longleftrightarrow & a_2 u_2^{-1} a_2 \\
s_1 & \longleftrightarrow & b_1 b_2 & & \\
s_2 & \longleftrightarrow & a_1 b_3 b_2 & & \\
s_3 & \longleftrightarrow & a_1 b_1^{-1} b_2 & & \\
s_4 & \longleftrightarrow & a_1 b_3^{-1} & & \\
s_5 & \longleftrightarrow & a_1 b_1 b_2^2 & & \\
 & & a_1 a_2^{-1} b_1^{-1} & \longleftrightarrow & u_1 \\
 & & a_2 a_1 b_1^{-1} & \longleftrightarrow & u_2 \\
 & & a_1^{-2} a_2 & \longleftrightarrow & u_3 \\
 & & a_1^{-1} a_2^{-1} a_1 b_1^{-1} & \longleftrightarrow & u_4
\end{array}$$

where

$$F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle,$$
$$F_5^{(v,s)} = \langle s_1, \ldots, s_5 \rangle,$$
$$F_2^{(h,a)} = \langle a_1, a_2 \rangle,$$
$$F_4^{(h,u)} = \langle u_1, \ldots, u_4 \rangle,$$
$$F_9^{(v,b)} = \langle b_3^{-1} b_1,\, b_2 b_1^2,\, b_3 b_1^2,\, b_1 b_2,\, b_2^{-1} b_3 b_1,\, b_1^{-1} b_2^2,\, b_1^{-2} b_3 b_2,\, b_1^{-3} b_2,\, b_1^{-2} b_2 b_1 \rangle,$$
$$F_9^{(v,s)} = \langle s_3 s_2^{-1},\, s_4 s_2^{-1},\, s_4^{-1} s_2^{-1},\, s_1,\, s_5 s_2^{-1},\, s_2 s_5,\, s_2^2,\, s_2 s_3,\, s_2 s_1 s_2^{-1} \rangle,$$
$$F_7^{(h,a)} = \langle a_1^2 a_2^{-1},\, a_1^{-1} a_2^{-2},\, a_2 a_1 a_2 a_1^{-1},\, a_1^{-2} a_2,\, a_1 a_2^{-2} a_1,\, a_1 a_2^2,\, a_1 a_2^{-1} a_1 a_2 \rangle,$$
$$F_7^{(h,u)} = \langle u_1 u_3 u_2^{-1},\, u_4 u_2^{-1},\, u_2 u_1^{-1},\, u_3,\, u_1^2,\, u_1 u_4,\, u_1 u_2 \rangle,$$

$$\begin{array}{rcl}
F_9^{(v,b)} & \overset{\cong}{\longleftrightarrow} & F_9^{(v,s)} \\
b_3^{-1} b_1 & \longleftrightarrow & s_3 s_2^{-1} \\
b_2 b_1^2 & \longleftrightarrow & s_4 s_2^{-1} \\
b_3 b_1^2 & \longleftrightarrow & s_4^{-1} s_2^{-1} \\
b_1 b_2 & \longleftrightarrow & s_1 \\
b_2^{-1} b_3 b_1 & \longleftrightarrow & s_5 s_2^{-1} \\
b_1^{-1} b_2^2 & \longleftrightarrow & s_2 s_5 \\
b_1^{-2} b_3 b_2 & \longleftrightarrow & s_2^2 \\
b_1^{-3} b_2 & \longleftrightarrow & s_2 s_3 \\
b_1^{-2} b_2 b_1 & \longleftrightarrow & s_2 s_1 s_2^{-1},
\end{array}$$

$$\begin{array}{rcl}
F_7^{(h,a)} & \overset{\cong}{\longleftrightarrow} & F_7^{(h,u)} \\
a_1^2 a_2^{-1} & \longleftrightarrow & u_1 u_3 u_2^{-1} \\
a_1^{-1} a_2^{-2} & \longleftrightarrow & u_4 u_2^{-1} \\
a_2 a_1 a_2 a_1^{-1} & \longleftrightarrow & u_2 u_1^{-1} \\
a_1^{-2} a_2 & \longleftrightarrow & u_3 \\
a_1 a_2^{-2} a_1 & \longleftrightarrow & u_1^2 \\
a_1 a_2^2 & \longleftrightarrow & u_1 u_4 \\
a_1 a_2^{-1} a_1 a_2 & \longleftrightarrow & u_1 u_2,
\end{array}$$

$$R(2,3) := \left\{ \begin{array}{ll} a_1 b_1 a_2 b_2, & a_1 b_2 a_2 b_1^{-1}, \\ \\ a_1 b_3 a_2^{-1} b_1, & a_1 b_3^{-1} a_1 b_2^{-1}, \\ \\ a_1 b_1^{-1} a_2^{-1} b_3, & a_2 b_3 a_2 b_2^{-1} \end{array} \right\}.$$

C Some lists

C.1 2-transitive $(6,6)$-groups

We study $(6,6)$-groups such that P_h, P_v are 2-transitive and give a complete list of the arising 4-tuples $(|P_h|, |P_v|, |P_h^{(2)}|, |P_v^{(2)}|)$. Without loss of generality, we may assume that $|P_h| \leq |P_v|$ and that $|P_h^{(2)}| \leq |P_v^{(2)}|$ if $|P_h| = |P_v|$. By Table 1, there are only four 2-transitive subgroups of S_6: $\mathrm{PSL}_2(5)$, $\mathrm{PGL}_2(5)$, A_6 and S_6 of order 60, 120, 360 and 720 respectively. Given $P_\bullet \in \{P_h, P_v\}$, the maximal possible value for $|P_\bullet^{(2)}|$ is $|P_\bullet|(|P_\bullet|/6)^6$. If this maximum is attained, the value of $|P_\bullet^{(2)}|$ is marked in the list with "$*$" on the right side. Observe that in the case $P_\bullet = A_6$ the value of $|P_\bullet^{(2)}|$ is always maximal (this is not very surprising by [15, Proposition 3.3.1]).

$\lvert P_h\rvert$	$\lvert P_v\rvert$	$\lvert P_h^{(2)}\rvert$	$\lvert P_v^{(2)}\rvert$
60	60	937500	937500
60	60	937500	60000000 $*$
60	120	7500	15000
60	120	937500	60000000
60	120	937500	120000000
60	120	937500	1920000000
60	120	30000000	1875000
60	120	30000000	60000000
60	120	30000000	1920000000
60	120	60000000 $*$	60000000
60	120	60000000 $*$	120000000
60	120	60000000 $*$	7680000000 $*$
60	360	937500	16796160000000 $*$
60	360	30000000	16796160000000 $*$
60	360	60000000 $*$	16796160000000 $*$
60	720	7500	1074954240000000
60	720	937500	33592320000000
60	720	937500	1074954240000000
60	720	937500	2149908480000000 $*$
60	720	1875000	1074954240000000
60	720	30000000	33592320000000
60	720	30000000	1074954240000000
60	720	30000000	2149908480000000 $*$
60	720	60000000 $*$	33592320000000
60	720	60000000 $*$	67184640000000
60	720	60000000 $*$	1074954240000000
60	720	60000000 $*$	2149908480000000 $*$
120	120	15000	15000
120	120	1875000	60000000
120	120	60000000	60000000
120	120	60000000	1920000000
120	120	60000000	3840000000
120	120	1920000000	1920000000
120	120	1920000000	7680000000 $*$
120	120	3840000000	7680000000 $*$

120	360	1875000		16796160000000	∗
120	360	60000000		16796160000000	∗
120	360	120000000		16796160000000	∗
120	360	1920000000		16796160000000	∗
120	360	3840000000		16796160000000	∗
120	360	7680000000	∗	16796160000000	∗
120	720	1875000		33592320000000	
120	720	1875000		1074954240000000	
120	720	60000000		33592320000000	
120	720	60000000		67184640000000	
120	720	60000000		1074954240000000	
120	720	60000000		2149908480000000	∗
120	720	120000000		33592320000000	
120	720	120000000		1074954240000000	
120	720	120000000		2149908480000000	∗
120	720	1920000000		33592320000000	
120	720	1920000000		67184640000000	
120	720	1920000000		1074954240000000	
120	720	1920000000		2149908480000000	∗
120	720	3840000000		33592320000000	
120	720	3840000000		67184640000000	
120	720	3840000000		1074954240000000	
120	720	3840000000		2149908480000000	∗
120	720	7680000000	∗	33592320000000	
120	720	7680000000	∗	1074954240000000	
120	720	7680000000	∗	2149908480000000	∗
360	360	16796160000000	∗	16796160000000	∗
360	720	16796160000000	∗	33592320000000	
360	720	16796160000000	∗	67184640000000	
360	720	16796160000000	∗	1074954240000000	
360	720	16796160000000	∗	2149908480000000	∗
720	720	33592320000000		3359232000000	
720	720	33592320000000		67184640000000	
720	720	33592320000000		1074954240000000	
720	720	33592320000000		2149908480000000	∗
720	720	67184640000000		1074954240000000	
720	720	67184640000000		2149908480000000	∗
720	720	1074954240000000		1074954240000000	
720	720	1074954240000000		2149908480000000	∗
720	720	2149908480000000	∗	2149908480000000	∗

C.2 $(4,4)$-groups

In the list below, we classify all $(4,4)$-groups by the permutation isomorphism types of P_h and P_v and by Γ^{ab} (only assuming $|P_h| \leq |P_v|$). We use the following notation:
2_1: group of order 2, permutation isomorphic to $\langle(1,2)\rangle < S_4$,
2_2: group of order 2, permutation isomorphic to $\langle(1,2)(3,4)\rangle$,
4_1: group of order 4, isomorphic to $\mathbb{Z}_2^2$, permutation isomorphic to $\langle(1,2),(3,4)\rangle$,
4_2: group of order 4, isomorphic to $\mathbb{Z}_2^2$, permutation isomorphic to $\langle(1,2)(3,4),(1,3)(2,4)\rangle$,
D_4: dihedral group of order 8.
$\mathrm{t}(P_\bullet)$ denotes the transitivity of the group $P_\bullet \in \{P_h, P_v\}$ on $\{1,2,3,4\}$.
?N? means that Γ is possibly irreducible.

Ex	P_h	P_v	$\mathrm{t}(P_h)$	$\mathrm{t}(P_v)$	reducible	Γ^{ab}
	1	1	0	0	Y	$\mathbb{Z}^4$
	1	2_1	0	0	Y	$\mathbb{Z}^3 \times \mathbb{Z}_2$
	1	2_2	0	0	Y	$\mathbb{Z}^3$
	1	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
	1	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
	1	4_1	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
	1	4_2	0	1	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
	1	D_4	0	1	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
	2_1	2_1	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
	2_1	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
	2_1	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_4$
	2_1	2_2	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2^3$
	2_2	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
	2_2	2_2	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
	2_1	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2^2$
	2_2	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_8$
	2_2	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2^2$
	2_1	4_1	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2^3$
	2_1	4_2	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
	2_2	4_1	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
	2_2	4_1	0	0	Y	$\mathbb{Z}^2$
	2_2	4_2	0	1	Y	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$
	2_1	D_4	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2^2$
	2_1	D_4	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
	2_2	A_4	0	2	Y	$\mathbb{Z} \times \mathbb{Z}_2$
26	$\mathbb{Z}_4$	$\mathbb{Z}_4$	1	1	Y	$\mathbb{Z}_4 \times \mathbb{Z}_8$
	$\mathbb{Z}_4$	4_1	1	0	Y	$\mathbb{Z} \times \mathbb{Z}_4$
30	4_1	4_1	0	0	Y	$\mathbb{Z}_2^4$
	4_1	D_4	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2$
	4_1	D_4	0	1	Y	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
34	D_4	A_4	1	2	?N?	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
35	S_4	S_4	4	4	?N?	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$

C.3 $(4,6)$-groups

The groups P_h and P_v in the next list are classified only up to isomorphism (*not* up to permutation isomorphism) and up to their transitivity. 36 denotes a group of order 36 permutation isomorphic to $\langle(1,2,3),(1,4,2,5)(3,6)\rangle$ and 72 denotes a group of order 72 permutation isomorphic to $\langle(1,2,3),(1,2),(1,4)(2,5)(3,6)\rangle$. ??Y?? means that we cannot exclude the existence of a reducible example.

Ex	P_h	P_v	t(P_h)	t(P_v)	reducible
	1	1	0	0	Y
	1	$\mathbb{Z}_2$	0	0	Y
	1	$\mathbb{Z}_3$	0	0	Y
	1	$\mathbb{Z}_4$	0	0	Y
	1	$\mathbb{Z}_2^2$	0	0	Y
	1	S_3	0	0	Y
	1	S_3	0	1	Y
	1	$\mathbb{Z}_6$	0	1	Y
	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	0	0	Y
	1	D_4	0	0	Y
	1	A_4	0	1	Y
	1	$\mathbb{Z}_2 \times S_3$	0	1	Y
	1	S_4	0	1	Y
	1	$\mathbb{Z}_2 \times A_4$	0	1	Y
	1	$\mathbb{Z}_2 \times S_4$	0	1	Y
	$\mathbb{Z}_2$	1	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_3$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	0	0	Y
	$\mathbb{Z}_2$	S_3	0	0	Y
	$\mathbb{Z}_2$	S_3	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_6$	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2$	D_4	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_3^2$	0	0	Y
	$\mathbb{Z}_2$	A_4	0	0	Y
	$\mathbb{Z}_2$	A_4	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times S_3$	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_3 \times S_3$	0	1	Y
	$\mathbb{Z}_2$	S_4	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times A_4$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times A_4$	0	1	Y
	$\mathbb{Z}_2$	36	0	1	Y
	$\mathbb{Z}_2$	$S_3 \times S_3$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times S_4$	0	1	Y
	$\mathbb{Z}_2$	$\mathrm{PSL}_2(5)$	0	2	Y
	$\mathbb{Z}_2$	$\mathrm{PGL}_2(5)$	0	3	Y
	$\mathbb{Z}_2$	A_6	0	4	Y

	$\mathbb{Z}_2$	S_6	0	6	Y
	$\mathbb{Z}_4$	1	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_2$	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	1	0	Y
	$\mathbb{Z}_4$	S_3	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	0	Y
	$\mathbb{Z}_4$	D_4	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_3^2$	1	0	Y
	$\mathbb{Z}_4$	$S_3 \times S_3$	1	0	Y
	$\mathbb{Z}_2^2$	1	0	0	Y
	$\mathbb{Z}_2^2$	1	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	1	0	Y
	$\mathbb{Z}_2^2$	S_3	0	0	Y
25	$\mathbb{Z}_2^2$	S_3	0	0	?N?
	$\mathbb{Z}_2^2$	S_3	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2^2$	D_4	0	0	Y
	$\mathbb{Z}_2^2$	A_4	0	1	Y
	$\mathbb{Z}_2^2$	A_4	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times S_3$	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times S_3$	0	1	?N?
	$\mathbb{Z}_2^2$	S_4	0	1	Y
	$\mathbb{Z}_2^2$	S_4	0	1	?N?
23, 28	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times A_4$	0	1	Y
29	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times A_4$	1	0	Y
	$\mathbb{Z}_2^2$	36	0	1	?N?
12	$\mathbb{Z}_2^2$	$S_3 \times S_3$	0	0	?N?
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times S_4$	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times S_4$	0	1	?N?
	$\mathbb{Z}_2^2$	$\mathrm{PSL}_2(5)$	0	2	?N?
	$\mathbb{Z}_2^2$	$\mathrm{PGL}_2(5)$	0	3	?N?
	$\mathbb{Z}_2^2$	S_6	0	6	N
	D_4	1	1	0	Y
	D_4	$\mathbb{Z}_2$	1	0	Y
	D_4	$\mathbb{Z}_3$	1	0	Y
	D_4	$\mathbb{Z}_4$	1	0	Y
	D_4	$\mathbb{Z}_2^2$	1	0	Y
	D_4	S_3	1	0	Y

	D_4	S_3	1	0	?N?
	D_4	S_3	1	1	Y
	D_4	$\mathbb{Z}_6$	1	1	Y
	D_4	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	0	Y
	D_4	D_4	1	0	Y
	D_4	$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	0	?N?
	D_4	A_4	1	0	Y
	D_4	A_4	1	0	?N?
	D_4	A_4	1	1	Y
	D_4	S_4	1	1	Y
	D_4	S_4	1	1	?N?
	D_4	$\mathbb{Z}_2 \times A_4$	1	0	Y
	D_4	$\mathbb{Z}_2 \times A_4$	1	0	?N?
	D_4	$\mathbb{Z}_2 \times A_4$	1	1	Y
	D_4	36	1	1	?N?
	D_4	$S_3 \times S_3$	1	0	?N?
	D_4	$\mathbb{Z}_2 \times S_4$	1	1	?N?
	D_4	$\mathrm{PSL}_2(5)$	1	2	?N?
	D_4	$\mathrm{PGL}_2(5)$	1	3	N
	D_4	$\mathrm{PGL}_2(5)$	1	3	??Y??
22	D_4	A_6	1	4	N
	D_4	S_6	1	6	N
	A_4	$\mathbb{Z}_2$	2	0	Y
	A_4	$\mathbb{Z}_2^2$	2	0	Y
	A_4	S_3	2	0	?N?
	A_4	D_4	2	0	?N?
	A_4	$\mathbb{Z}_2 \times S_3$	2	1	?N?
	A_4	S_4	2	1	?N?
	A_4	36	2	1	?N?
	A_4	$S_3 \times S_3$	2	0	?N?
	A_4	$\mathbb{Z}_2 \times S_4$	2	1	?N?
	A_4	S_6	2	6	N
	S_4	$\mathbb{Z}_2$	4	0	Y
	S_4	$\mathbb{Z}_4$	4	0	Y
	S_4	$\mathbb{Z}_2^2$	4	0	Y
	S_4	S_3	4	0	N
	S_4	S_3	4	0	??Y??
	S_4	$\mathbb{Z}_2 \times \mathbb{Z}_4$	4	0	Y
	S_4	D_4	4	0	Y
	S_4	D_4	4	0	?N?
	S_4	$\mathbb{Z}_3 \times \mathbb{Z}_3$	4	0	?N?
	S_4	S_4	4	0	?N?
	S_4	S_4	4	1	N
	S_4	S_4	4	1	??Y??
	S_4	$S_3 \times S_3$	4	0	N
	S_4	$S_3 \times S_3$	4	0	??Y??

	S_4	$\mathbb{Z}_2 \times S_4$	4	0	?N?
	S_4	$\mathbb{Z}_2 \times S_4$	4	1	N
	S_4	$\mathbb{Z}_2 \times S_4$	4	1	??Y??
	S_4	$\mathrm{PSL}_2(5)$	4	2	N
	S_4	$\mathrm{PSL}_2(5)$	4	2	??Y??
	S_4	72	4	1	?N?
52	S_4	$\mathrm{PGL}_2(5)$	4	3	N
	S_4	$\mathrm{PGL}_2(5)$	4	3	??Y??
	S_4	A_6	4	4	N
	S_4	S_6	4	6	N

C.4 Some abelianized (A_{2m}, A_{2n})-groups

We classify some (A_{2m}, A_{2n})-groups Γ by their abelianization Γ^{ab} and by the size of $P_h^{(2)}$ and $P_v^{(2)}$ (we restrict to $2 \leq m \leq n$; $m + n \leq 8$). If $P_h^{(2)}$ is not maximal (this can only happen for $2m = 4$), then we give the number $12 \cdot 3^4/|P_h^{(2)}|$. The list is complete for $(2m, 2n) = (6, 6)$ and $(2m, 2n) = (4, 8)$. There are no (A_4, A_4)- and (A_4, A_6)-groups.

Ex	$2m$	$2n$	$P_h^{(2)}$ max.	$P_v^{(2)}$ max.	$\lvert\Gamma^{ab}\rvert$	Γ^{ab}
	4	8	Y	Y	4	$\mathbb{Z}_2^2$
	4	10	Y	Y	4	$\mathbb{Z}_2^2$
	4	10	3	Y	4	$\mathbb{Z}_2^2$
	4	10	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	10	3	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	10	Y	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	4	10	3	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	4	10	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	4	10	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	10	3	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	10	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	4	10	Y	Y	24	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$
	4	10	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	4	12	Y	Y	4	$\mathbb{Z}_2^2$
	4	12	3	Y	4	$\mathbb{Z}_2^2$
	4	12	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	12	3	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	12	Y	Y	8	$\mathbb{Z}_2^3$
	4	12	3	Y	8	$\mathbb{Z}_2^3$
	4	12	Y	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	4	12	3	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	4	12	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	12	3	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	12	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	4	12	Y	Y	20	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$
	4	12	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	4	12	Y	Y	24	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$
	4	12	Y	Y	28	$\mathbb{Z}_2^2 \times \mathbb{Z}_7$
	4	12	Y	Y	32	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
	4	12	3	Y	32	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
	4	12	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	4	12	Y	Y	40	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
	4	12	Y	Y	40	$\mathbb{Z}_2^3 \times \mathbb{Z}_5$
	4	12	Y	Y	48	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8$
1	6	6	Y	Y	4	$\mathbb{Z}_2^2$
	6	6	Y	Y	8	$\mathbb{Z}_2^3$
	6	6	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	6	6	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	6	6	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	6	6	Y	Y	28	$\mathbb{Z}_2^2 \times \mathbb{Z}_7$

2	6	6	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	6	8	Y	Y	4	$\mathbb{Z}_2^2$
	6	8	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	6	8	Y	Y	8	$\mathbb{Z}_2^3$
	6	8	Y	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	6	8	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	6	8	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	6	8	Y	Y	16	$\mathbb{Z}_2^4$
	6	8	Y	Y	20	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$
	6	8	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	6	8	Y	Y	24	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$
	6	8	Y	Y	28	$\mathbb{Z}_2^2 \times \mathbb{Z}_7$
	6	8	Y	Y	32	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
	6	8	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	6	8	Y	Y	36	$\mathbb{Z}_2^2 \times \mathbb{Z}_9$
	6	8	Y	Y	40	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
	6	8	Y	Y	40	$\mathbb{Z}_2^3 \times \mathbb{Z}_5$
	6	8	Y	Y	48	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	6	8	Y	Y	60	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
	6	8	Y	Y	80	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
	6	10	Y	Y	4	$\mathbb{Z}_2^2$
	6	10	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	6	10	Y	Y	8	$\mathbb{Z}_2^3$
	6	10	Y	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	6	10	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	6	10	Y	Y	16	$\mathbb{Z}_4^2$
	6	10	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	6	10	Y	Y	20	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$
	6	10	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	6	10	Y	Y	24	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$
	6	10	Y	Y	28	$\mathbb{Z}_2^2 \times \mathbb{Z}_7$
	6	10	Y	Y	40	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
	6	10	Y	Y	40	$\mathbb{Z}_2^3 \times \mathbb{Z}_5$
	6	10	Y	Y	108	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$
	8	8	Y	Y	4	$\mathbb{Z}_2^2$
	8	8	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	8	8	Y	Y	8	$\mathbb{Z}_2^3$
	8	8	Y	Y	12	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$
	8	8	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	8	8	Y	Y	16	$\mathbb{Z}_4^2$
	8	8	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	8	8	Y	Y	16	$\mathbb{Z}_2^4$
	8	8	Y	Y	20	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$
	8	8	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
	8	8	Y	Y	24	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$
	8	8	Y	Y	28	$\mathbb{Z}_2^2 \times \mathbb{Z}_7$

C.5 Embedding Example 13 into primitive $(10, 10)$-groups

We embed the non-residually finite $(8, 6)$-complex of Example 13 into $(10, 10)$-complexes X such that P_h and P_v are primitive permutation groups. Let $w := a_2 a_1^{-1} a_3 a_4^{-1}$. In all examples Γ in the subsequent list, the normal subgroup $\langle\langle w \rangle\rangle_\Gamma$ has finite index in Γ, in particular, by Lemma 29,

$$\langle\langle w \rangle\rangle_\Gamma = \bigcap_{N \overset{\text{f.i.}}{\triangleleft} \Gamma} N.$$

If two rows are exactly the same, then the quotients $\Gamma / \langle\langle w \rangle\rangle_\Gamma$ are non-isomorphic non-abelian groups of the same finite order. The (A_{10}, A_{10})-groups are the same as in Table 8.

P_h	P_v	Γ^{ab}	$\lvert\Gamma^{ab}\rvert$	$[\Gamma : \langle\langle w \rangle\rangle_\Gamma]$
S_6	A_{10}	$[2, 2]$	4	4
S_6	S_{10}	$[2, 2]$	4	4
$\mathrm{P\Gamma L}_2(9)$	A_{10}	$[2, 2]$	4	4
$\mathrm{P\Gamma L}_2(9)$	S_{10}	$[2, 2]$	4	4
$\mathrm{P\Gamma L}_2(9)$	S_{10}	$[2, 4]$	8	8
$\mathrm{P\Gamma L}_2(9)$	S_{10}	$[2, 2, 2]$	8	8
A_{10}	A_{10}	$[2, 2]$	4	4
A_{10}	A_{10}	$[2, 4]$	8	8
A_{10}	A_{10}	$[2, 2, 2]$	8	8
A_{10}	A_{10}	$[2, 2, 3]$	12	12
A_{10}	A_{10}	$[2, 2, 4]$	16	16
A_{10}	A_{10}	$[2, 8]$	16	16
A_{10}	A_{10}	$[2, 2, 5]$	20	20
A_{10}	A_{10}	$[2, 3, 4]$	24	24
A_{10}	A_{10}	$[2, 2, 2, 3]$	24	24
A_{10}	A_{10}	$[2, 2, 8]$	32	32
A_{10}	A_{10}	$[2, 4, 5]$	40	40
A_{10}	S_{10}	$[2, 2]$	4	4
A_{10}	S_{10}	$[2, 4]$	8	8
A_{10}	S_{10}	$[2, 2, 2]$	8	8
A_{10}	S_{10}	$[2, 2, 2]$	8	< 16
A_{10}	S_{10}	$[2, 2, 3]$	12	12
A_{10}	S_{10}	$[2, 8]$	16	16
A_{10}	S_{10}	$[4, 4]$	16	16
A_{10}	S_{10}	$[2, 2, 4]$	16	16
A_{10}	S_{10}	$[2, 2, 5]$	20	20
A_{10}	S_{10}	$[2, 3, 4]$	24	24
A_{10}	S_{10}	$[2, 2, 2, 3]$	24	24
A_{10}	S_{10}	$[2, 2, 7]$	28	28
A_{10}	S_{10}	$[2, 2, 8]$	32	32
A_{10}	S_{10}	$[2, 16]$	32	32
A_{10}	S_{10}	$[2, 4, 5]$	40	40
A_{10}	S_{10}	$[2, 2, 2, 5]$	40	40
A_{10}	S_{10}	$[2, 3, 8]$	48	48
S_{10}	A_{10}	$[2, 2]$	4	4

S_{10}	A_{10}	$[2,4]$	8	8
S_{10}	A_{10}	$[2,2,2]$	8	8
S_{10}	A_{10}	$[2,2,2]$	8	<16
S_{10}	A_{10}	$[2,2,2]$	8	<16
S_{10}	A_{10}	$[2,2,3]$	12	12
S_{10}	A_{10}	$[2,2,4]$	16	16
S_{10}	A_{10}	$[2,8]$	16	16
S_{10}	A_{10}	$[4,4]$	16	16
S_{10}	A_{10}	$[2,2,5]$	20	20
S_{10}	A_{10}	$[2,3,4]$	24	24
S_{10}	A_{10}	$[2,2,2,3]$	24	24
S_{10}	A_{10}	$[2,2,7]$	28	28
S_{10}	A_{10}	$[2,2,8]$	32	32
S_{10}	A_{10}	$[2,2,9]$	36	36
S_{10}	A_{10}	$[2,2,3,3]$	36	36
S_{10}	A_{10}	$[2,4,5]$	40	40
S_{10}	A_{10}	$[2,2,11]$	44	44
S_{10}	A_{10}	$[2,4,7]$	56	56
S_{10}	A_{10}	$[2,32]$	64	64
S_{10}	S_{10}	$[2,2]$	4	4
S_{10}	S_{10}	$[2,4]$	8	8
S_{10}	S_{10}	$[2,2,2]$	8	8
S_{10}	S_{10}	$[2,2,2]$	8	<16
S_{10}	S_{10}	$[2,2,2]$	8	<16
S_{10}	S_{10}	$[2,2,3]$	12	12
S_{10}	S_{10}	$[2,8]$	16	16
S_{10}	S_{10}	$[2,2,4]$	16	16
S_{10}	S_{10}	$[2,2,4]$	16	<32
S_{10}	S_{10}	$[2,2,4]$	16	<32
S_{10}	S_{10}	$[2,2,4]$	16	<32
S_{10}	S_{10}	$[4,4]$	16	16
S_{10}	S_{10}	$[2,2,5]$	20	20
S_{10}	S_{10}	$[2,3,4]$	24	24
S_{10}	S_{10}	$[2,2,2,3]$	24	24
S_{10}	S_{10}	$[2,2,2,3]$	24	<48
S_{10}	S_{10}	$[2,2,7]$	28	28
S_{10}	S_{10}	$[2,16]$	32	32
S_{10}	S_{10}	$[2,2,8]$	32	32
S_{10}	S_{10}	$[2,4,4]$	32	32
S_{10}	S_{10}	$[4,8]$	32	32
S_{10}	S_{10}	$[2,2,9]$	36	36
S_{10}	S_{10}	$[2,2,3,3]$	36	36
S_{10}	S_{10}	$[2,4,5]$	40	40
S_{10}	S_{10}	$[2,2,2,5]$	40	40
S_{10}	S_{10}	$[2,2,11]$	44	44
S_{10}	S_{10}	$[2,3,8]$	48	48

S_{10}	S_{10}	$[2, 2, 3, 4]$	48	48
S_{10}	S_{10}	$[2, 2, 13]$	52	52
S_{10}	S_{10}	$[2, 4, 7]$	56	56
S_{10}	S_{10}	$[2, 2, 3, 5]$	60	60
S_{10}	S_{10}	$[2, 32]$	64	64
S_{10}	S_{10}	$[2, 4, 9]$	72	72
S_{10}	S_{10}	$[2, 2, 19]$	76	76
S_{10}	S_{10}	$[2, 5, 8]$	80	80
S_{10}	S_{10}	$[2, 4, 11]$	88	88
S_{10}	S_{10}	$[2, 2, 25]$	100	100
S_{10}	S_{10}	$[2, 2, 5, 5]$	100	100
S_{10}	S_{10}	$[2, 4, 13]$	104	104

D GAP programs

In this section, we want to describe some GAP-programs ([28]), which have led to the construction of most groups in this paper.

D.1 Theory and ideas

Our strategy can be resumed as follows:

Step 1: Describe a $(2m, 2n)$-complex X in a way which is manageable for a computer. We have adopted an idea due to ???, who writes X as a pair of integer valued matrices (lists of lists) ???
A and B.

Step 2: Given small m, n, generate *all* pairs (A, B) corresponding to a $(2m, 2n)$-complex. Given large m, n, generate *randomly* many pairs (A, B) corresponding to a $(2m, 2n)$-complex.

Step 3: Starting from a constructed pair (A, B), provide additional programs which compute the local groups $P_h^{(k)}$, $P_v^{(k)}$ (for k small) and a finite presentation of Γ. Then apply the powerful GAP-tools for finite permutation groups to look for examples with interesting local groups and/or use GAP-commands like AbelianInvariants() and LowIndexSubgroupsFpGroup() to get some information on the normal subgroup structure of the infinite group Γ.

For instance, we have immediately found in this way an irreducible (A_6, A_6)-group Γ with $[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 perfect (see Example 1).

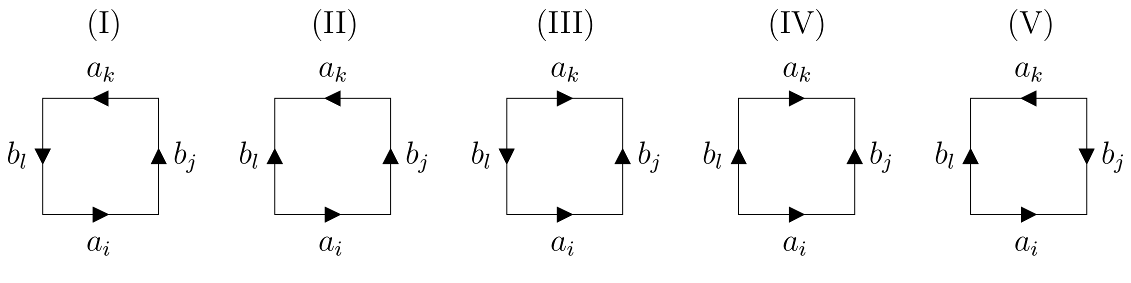
Coming back to Step 1, we will define for given $m, n \in \mathbb{N}$ an injective map

$$\begin{aligned} \varphi_{m,n} : \mathcal{X}_{2m,2n} &\to \mathrm{Mat}(2m, 2n, \{1, \ldots, 2m\}) \times \mathrm{Mat}(2m, 2n, \{1, \ldots, 2n\}) \\ X &\mapsto \varphi_{m,n}(X) = (A, B) \end{aligned}$$

where $\mathcal{X}_{2m,2n}$ denotes the set of $(2m, 2n)$-complexes and $X \in \mathcal{X}_{2m,2n}$ is given by its mn geometric squares and where $\mathrm{Mat}(2m, 2n, \{1, \ldots, 2m\})$ denotes the set of $(2m \times 2n)$-matrices with entries in the set $\{1, \ldots, 2m\}$. Each geometric square $aba'b'$ of X can be represented by expressions of the form

$$aba'b', \quad a'b'ab, \quad a^{-1}b'^{-1}a'^{-1}b^{-1}, \quad a'^{-1}b^{-1}a^{-1}b'^{-1}.$$

To define $\varphi_{m,n}$, note that one (or two) of these expressions has one of the five types (I)-(V) illustrated in Figure 18, for suitable $i, k \in \{1, \ldots, m\}$ and $j, l \in \{1, \ldots, n\}$. It is easy to check



$a_i b_j a_k b_l$ $\qquad$ $a_i b_j a_k b_l^{-1}$ $\qquad$ $a_i b_j a_k^{-1} b_l$ $\qquad$ $a_i b_j a_k^{-1} b_l^{-1}$ $\qquad$ $a_i b_j^{-1} a_k b_l^{-1}$

Figure 18: Possible types of a geometric square

that each geometric square has a unique type. We now define $\varphi_{m,n}$ for each possible type of geometric squares, using the following notation for the "inverses": $\bar{i} := 2m+1-i$, $\bar{k} := 2m+1-k$,

$\bar{j} := 2n+1-j$, $\bar{l} := 2n+1-l$.

Type (I) $(a_i b_j a_k b_l)$

$$\begin{aligned} A_{ij} &:= \bar{k}, \quad B_{ij} := \bar{l} \\ A_{kl} &:= \bar{i}, \quad B_{kl} := \bar{j} \\ A_{\bar{i}\bar{l}} &:= k, \quad B_{\bar{i}\bar{l}} := j \\ A_{\bar{k}\bar{j}} &:= i, \quad B_{\bar{k}\bar{j}} := l. \end{aligned}$$

Type (II) $(a_i b_j a_k b_l^{-1})$

$$\begin{aligned} A_{ij} &:= \bar{k}, \quad B_{ij} := l \\ A_{k\bar{l}} &:= \bar{i}, \quad B_{k\bar{l}} := \bar{j} \\ A_{\bar{i}l} &:= k, \quad B_{\bar{i}l} := j \\ A_{\bar{k}\bar{j}} &:= i, \quad B_{\bar{k}\bar{j}} := \bar{l}. \end{aligned}$$

Type (III) $(a_i b_j a_k^{-1} b_l)$

$$\begin{aligned} A_{ij} &:= k, \quad B_{ij} := \bar{l} \\ A_{k\bar{j}} &:= i, \quad B_{k\bar{j}} := l \\ A_{\bar{i}\bar{l}} &:= \bar{k}, \quad B_{\bar{i}\bar{l}} := j \\ A_{\bar{k}l} &:= \bar{i}, \quad B_{\bar{k}l} := \bar{j}. \end{aligned}$$

Type (IV) $(a_i b_j a_k^{-1} b_l^{-1})$

$$\begin{aligned} A_{ij} &:= k, \quad B_{ij} := l \\ A_{k\bar{j}} &:= i, \quad B_{k\bar{j}} := \bar{l} \\ A_{\bar{i}l} &:= \bar{k}, \quad B_{\bar{i}l} := j \\ A_{\bar{k}\bar{l}} &:= \bar{i}, \quad B_{\bar{k}\bar{l}} := \bar{j}. \end{aligned}$$

Type (V) $(a_i b_j^{-1} a_k b_l^{-1})$

$$\begin{aligned} A_{i\bar{j}} &:= \bar{k}, \quad B_{i\bar{j}} := l \\ A_{k\bar{l}} &:= \bar{i}, \quad B_{k\bar{l}} := j \\ A_{\bar{i}l} &:= k, \quad B_{\bar{i}l} := \bar{j} \\ A_{\bar{k}j} &:= i, \quad B_{\bar{k}j} := \bar{l}. \end{aligned}$$

Thus, each geometric square of X defines exactly four entries in A and B which describe the corresponding four edges in the link $Lk(x)$. In case of type (I) and (V) two choices are possible, since as geometric squares we have the equalities $a_i b_j a_k b_l = a_k b_l a_i b_j$ and $a_i b_j^{-1} a_k b_l^{-1} = a_k b_l^{-1} a_i b_j^{-1}$ respectively, but the definition of $\varphi_{m,n}$ is independent of this choice. This proves that $\varphi_{m,n}$ is well-defined.

We illustrate this definition in Table 31 in the case of Example 1 given by its nine geometric squares (see also Figure 19)

$$R(3,3) := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, & a_1 b_3 a_2 b_2^{-1}, \\ & & \\ a_1 b_3^{-1} a_3^{-1} b_2, & a_2 b_1 a_3^{-1} b_2^{-1}, & a_2 b_2 a_3^{-1} b_3^{-1}, \\ & & \\ a_2 b_3 a_3^{-1} b_1, & a_2 b_3^{-1} a_3 b_2, & a_2 b_1^{-1} a_3^{-1} b_1^{-1} \end{array} \right\}.$$

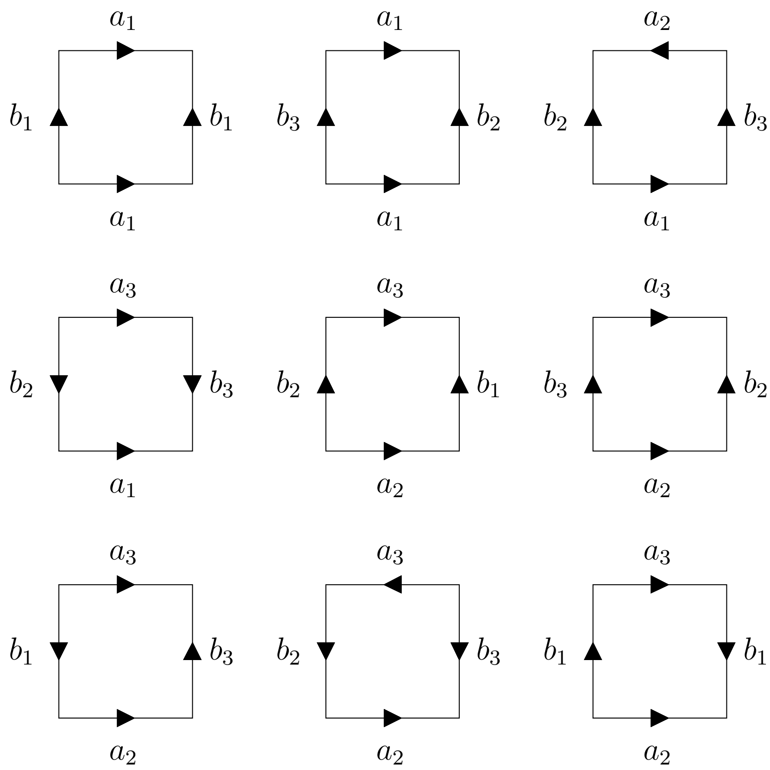


Figure 19: Example 1

Hence

$$A = \begin{pmatrix} 1 & 1 & 5 & 3 & 1 & 1 \\ 3 & 3 & 3 & 4 & 6 & 3 \\ 2 & 5 & 1 & 2 & 2 & 2 \\ 5 & 6 & 2 & 5 & 5 & 5 \\ 4 & 4 & 4 & 1 & 3 & 4 \\ 6 & 2 & 6 & 6 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \\ 6 & 3 & 2 & 1 & 4 & 5 \\ 4 & 3 & 2 & 5 & 6 & 1 \\ 6 & 1 & 2 & 5 & 4 & 3 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix}$$

or in a more compact form

Note that given $(A, B) \in \mathrm{im}(\varphi_{m,n})$, we can uniquely reconstruct $X = \varphi_{m,n}^{-1}(A, B)$ (reflecting the injectivity of $\varphi_{m,n}$).

Remark. By construction of $\varphi_{m,n}$, there are bijections between the following sets:

$$\{(A_{ij}, B_{ij})\}_{i=1,\dots,2m,\ j=1,\dots,2n} \cong \{1, \dots, 2m\} \times \{1, \dots, 2n\},$$

$$\{1, \dots 2m\} \cong \{A_{ij}\}_{i=1,\dots,2m} \text{ for any } j \in \{1, \dots, 2n\},$$

$$\{1, \dots 2n\} \cong \{B_{ij}\}_{j=1,\dots,2n} \text{ for any } i \in \{1, \dots, 2m\}.$$

The idea of Step 2 for small m, n is to start with $(2m \times 2n)$-matrices A and B consisting of 0-entries and "fill" them recursively with one geometric square (four non-zero entries in A and B) in each recursion step. This is done systematically, i.e. going through all potential geometric squares S. Of course, S has to satisfy several conditions, e.g. we want all potential new positions in A (and B) coming from S to be free (i.e. zeroes) and all potential new pairs of entries $(A_{\alpha\beta}, B_{\alpha\beta})$ coming from S are required to be new. If the candidate S does not satisfy these conditions, we try

geometric square	representative	type	A-entries	B-entries
$a_1b_1a_1^{-1}b_1^{-1}$	$a_1b_1a_1^{-1}b_1^{-1}$	(IV)	$A_{11}=1$	$B_{11}=1$
			$A_{16}=1$	$B_{16}=6$
			$A_{61}=6$	$B_{61}=1$
			$A_{66}=6$	$B_{66}=6$
$a_1b_2a_1^{-1}b_3^{-1}$	$a_1b_2a_1^{-1}b_3^{-1}$	(IV)	$A_{12}=1$	$B_{12}=3$
			$A_{15}=1$	$B_{15}=4$
			$A_{63}=6$	$B_{63}=2$
			$A_{64}=6$	$B_{64}=5$
$a_1b_3a_2b_2^{-1}$	$a_1b_3a_2b_2^{-1}$	(II)	$A_{13}=5$	$B_{13}=2$
			$A_{25}=6$	$B_{25}=4$
			$A_{62}=2$	$B_{62}=3$
			$A_{54}=1$	$B_{54}=5$
$a_1b_3^{-1}a_3^{-1}b_2$	$a_3b_3a_1^{-1}b_2^{-1}$	(IV)	$A_{33}=1$	$B_{33}=2$
			$A_{14}=3$	$B_{14}=5$
			$A_{42}=6$	$B_{42}=3$
			$A_{65}=4$	$B_{65}=4$
$a_2b_1a_3^{-1}b_2^{-1}$	$a_2b_1a_3^{-1}b_2^{-1}$	(IV)	$A_{21}=3$	$B_{21}=2$
			$A_{36}=2$	$B_{36}=5$
			$A_{52}=4$	$B_{52}=1$
			$A_{45}=5$	$B_{45}=6$
$a_2b_2a_3^{-1}b_3^{-1}$	$a_2b_2a_3^{-1}b_3^{-1}$	(IV)	$A_{22}=3$	$B_{22}=3$
			$A_{35}=2$	$B_{35}=4$
			$A_{53}=4$	$B_{53}=2$
			$A_{44}=5$	$B_{44}=5$
$a_2b_3a_3^{-1}b_1$	$a_2b_3a_3^{-1}b_1$	(III)	$A_{23}=3$	$B_{23}=6$
			$A_{34}=2$	$B_{34}=1$
			$A_{56}=4$	$B_{56}=3$
			$A_{41}=5$	$B_{41}=4$
$a_2b_3^{-1}a_3b_2$	$a_3b_2a_2b_3^{-1}$	(II)	$A_{32}=5$	$B_{32}=3$
			$A_{24}=4$	$B_{24}=5$
			$A_{43}=2$	$B_{43}=2$
			$A_{55}=3$	$B_{55}=4$
$a_2b_1^{-1}a_3^{-1}b_1^{-1}$	$a_3b_1a_2^{-1}b_1$	(III)	$A_{31}=2$	$B_{31}=6$
			$A_{26}=3$	$B_{26}=1$
			$A_{46}=5$	$B_{46}=1$
			$A_{51}=4$	$B_{51}=6$

Table 31: Definition of A and B for Example 1

the next one. The conditions guarantee that at the end a “full” (i.e. without zero entries) pair of matrices (A, B) indeed describes a $(2m, 2n)$-complex X, in particular having a complete bipartite link $Lk(x)$ as required.

D.2 The main program

Our GAP-program looks as follows: (comments in GAP start with the character #)

$\varphi_{3,3}(X)$	$1 \approx b_1$	$2 \approx b_2$	$3 \approx b_3$	$4 \approx b_3^{-1}$	$5 \approx b_2^{-1}$	$6 \approx b_1^{-1}$
$1 \approx a_1$	1/1	1/3	5/2	3/5	1/4	1/6
$2 \approx a_2$	3/2	3/3	3/6	4/5	6/4	3/1
$3 \approx a_3$	2/6	5/3	1/2	2/1	2/4	2/5
$4 \approx a_3^{-1}$	5/4	6/3	2/2	5/5	5/6	5/1
$5 \approx a_2^{-1}$	4/6	4/1	4/2	1/5	3/4	4/3
$6 \approx a_1^{-1}$	6/1	2/3	6/2	6/5	4/4	6/6

```
all := function(x1, x2, y1, y2)
# generates the list [[x1,y1],...,[x1,y2],...,[x2,y1],...,[x2,y2]]
local w, k, i, j;
w := [ ];
k := 1;
for i in [x1..x2] do
  for j in [y1..y2] do
    w[k] := [i,j];
    k := k+1;
  od;
od;
return w;
end;

test := function(M, N, q, r, s, t, cM, cN)
# checks candidate a_q*b_r*a_s^-1*b_t^-1
if (s = cM+1-q and t = cN+1-r) or
    M[s][cN+1-r] <> 0 or M[cM-q+1][t] <> 0 or M[cM+1-s][cN+1-t] <> 0 or
# M[q][r] <> 0 is tested in test2
    ForAny(all(1,cM,1,cN),
    v -> ([M[v[1]][v[2]],N[v[1]][v[2]]]
           in [[s,t], [q,cN+1-t], [cM+1-s,r], [cM+1-q,cN+1-r]]))
then
  return false;
else
  return true;
fi;
end;

part := function(x, y, z)
# we assume y <= z
# generates list [[1,1],...,[1,z],...,[x-1,1],...,[x-1,z],[x,1],...,,[x,y-1]]
local w, k, i1, i2, j;
w := [ ];
k := 1;
for i1 in [1..x-1] do
  for i2 in [1..z] do
    w[k] := [i1,i2];
    k := k+1;
  od;
```

```
od;
for j in [1..y-1] do
  w[k] := [x,j];
  k := k+1;
od;
return w;
end;

test2 := function(A, x, y, z)
# returns true if (x,y) is the first "free" position in A
if A[x][y] = 0 and
  ForAll(part(x,y,z), v -> A[v[1]][v[2]] <> 0)
then
  return true;
else
  return false;
fi;
end;

full := function(A)
# returns true if matrix A contains no 0
if ForAny(A, x -> 0 in x) then
  return false;
else
  return true;
fi;
end;

main := function(A, B)
# main program
local cA, cB, i, j, k, l, AA, BB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
for i in [1..cA/2] do
  for j in [1..cB] do
    if test2(A,i,j,cB) then # (i,j) is first free position in A
      for k in [1..cA] do
         for l in [1..cB] do
           if test(A,B,i,j,k,l,cA,cB) then # tests if a_i*b_j*a_k^-1*b_l^-1 is ok
             AA := StructuralCopy(A);
             BB := StructuralCopy(B);
             AA[i][j] := k;
             BB[i][j] := l;
             AA[k][cB-j+1] := i;
             BB[k][cB-j+1] := cB+1-l;
             AA[cA+1-i][l] := cA+1-k;
             BB[cA+1-i][l] := j;
             AA[cA+1-k][cB+1-l] := cA+1-i;
             BB[cA+1-k][cB+1-l] := cB+1-j;
```

```
          if full(AA) then # (AA,BB) now describes a (cA,cB)-complex
            # now we can check for conditions on AA, BB, e.g. like
            # if conditions(AA,BB) then Print(AA, " ", BB, "\n"); fi;
          else
            main(AA, BB); # recursive step
          fi;
        fi;
      od;
    od;
  fi;
 od;
od;
end;

# e.g. main(NullMat(4, 6), NullMat(4, 6)); generates all (4,6)-complexes
# e.g. main(C,D); where C, D describe any partial complex,
# i.e. some given geometric squares
```

This procedure can also be applied for large m and n, but the time required to finish (i.e. to generate *all* $(2m, 2n)$-complexes) grows very rapidly with increasing m and n. One reason for this is that the filling process needs mn recursion steps for each $(2m, 2n)$-complex but another reason is that the number of different $(2m, 2n)$-complexes becomes very large soon. This is illustrated in Table 32 (note that we do not claim that different examples are non-isomorphic groups).

m	n	mn	$\# X$
1	1	1	3
1	2	2	15
1	3	3	105
1	4	4	945
1	5	5	10395
2	2	4	541
2	3	6	35235
2	4	8	3690009
3	3	9	27712191

Table 32: Number of $(2m, 2n)$-complexes generated by our programs

Therefore, to get a better "distribution" of the examples, we have also written a program which randomly generates $(2m, 2n)$-complexes.

D.3 A random program

```
# the functions full(), all(), test(), part(), test2() are as before

Ma := function(m, n)
# generates (m x n)-matrix A, A[i][j] = i
local i, j, w;
w := NullMat(m,n);
for i in [1..m] do
 for j in [1..n] do
```

```
    w[i][j] := i;
  od;
od;
return w;
end;

Mb := function(m, n)
# generates (m x n)-matrix A, A[i][j] = j
local i, j, w;
w := NullMat(m,n);
for i in [1..m] do
  for j in [1..n] do
    w[i][j] := j;
  od;
od;
return w;
end;

out := [];

rdm := function(A, B, p)
local cA, cB, i, j, k, l, AA, BB, kl, pp, z;
z := 0;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
for i in [1..cA/2] do
  for j in [1..cB] do
    if test2(A,i,j,cB) then
      repeat kl := Random(p); # p:available edges in link
              z := z+1;
              # z counts number of attempts, here we set the maximal number to 30,
              # but it can be chosen larger or smaller if needed
      until test(A,B,i,j,kl[1],kl[2],cA,cB) or z = 30;
      AA := StructuralCopy(A);
      BB := StructuralCopy(B);
      if z < 30 then # test ok
         AA[i][j] := kl[1];
         BB[i][j] := kl[2];
         AA[kl[1]][cB-j+1] := i;
         BB[kl[1]][cB-j+1] := cB+1-kl[2];
         AA[cA+1-i][kl[2]] := cA+1-kl[1];
         BB[cA+1-i][kl[2]] := j;
         AA[cA+1-kl[1]][cB+1-kl[2]] := cA+1-i;
         BB[cA+1-kl[1]][cB+1-kl[2]] := cB+1-j;
         pp := StructuralCopy(p);
         RemoveSet(pp,kl);
         RemoveSet(pp,[i,cB+1-kl[2]]);
         RemoveSet(pp,[cA+1-kl[1],j]);
         RemoveSet(pp,[cA+1-i,cB+1-j]); # removes used edges in link
```

```
        if full(AA) then
          out := StructuralCopy([AA,BB,cA,cB]);
        else
          rdm(AA, BB, pp);
        fi;
     fi;
   fi;
 od;
od;
return out;
end;

slc := function(aa,bb)
local res;
repeat out := [Ma(aa,bb),Mb(aa,bb),aa,bb];
       res := rdm(NullMat(aa, bb), NullMat(aa, bb), all(1,aa,1,bb));
until # conditions(res[1],res[2]); whatever we want to check
Print(res[1],"\n",res[2],"\n");
end;

# e.g. slc(6,6); generates now randomly a (6,6)-complex
# satisfying additional conditions
```

One nice feature of both programs is that we can start with any k given geometric squares ($0 \leq k < mn$) and generate all (some) $(2m, 2n)$-complexes containing these k geometric squares. This was very useful in Section 3, where we have embedded for instance non-residually finite examples in virtually simple $(2m, 2n)$-complexes.

D.4 Computing the local groups

For Step 3 we have written programs which compute the local groups $P_h^{(k)}$, $P_v^{(k)}$ for k small enough. Here are the programs for $k = 1$ and $k = 2$ (the programs for $k \geq 3$ become more complicated with increasing k, but we do not need any new ideas). Moreover, we give the program to compute K_h for $m = 3$.

```
PhPerm := function(j, cA, A)
# generates permutation in P_h induced by b_j, i.e. $\rho_v(b_j)$
local v, i;
v := [ ];
for i in [1..cA] do
 v[i] := cA+1-A[cA-i+1][j];
od;
return PermList(v);
end;

Ph := function(A)
# generates P_h as a permutation group
local p, j, cA, cB;
cA := DimensionsMat(A)[1];
```

```
cB := DimensionsMat(A)[2];
p := [ ];
for j in [1..cB/2] do
  p[j] := PhPerm(j,cA,A);
od;
return Group(p,());
end;

PvPerm := function(i, cA, cB, B)
# generates permutation in P_v induced by a_i, i.e. ρ_h(a_i)
local w, j;
w := [ ];
for j in [1..cB] do
  w[j] := B[cA-i+1][j];
od;
return PermList(w);
end;

Pv := function(B)
# generates P_v
local p, i, cA, cB;
cA := DimensionsMat(B)[1];
cB := DimensionsMat(B)[2];
p := [ ];
for i in [1..cA/2] do
  p[i] := PvPerm(i,cA,cB,B);
od;
return Group(p,());
end;

indx := function(v, x)
# returns index of first appearance of x in vector v
local i;
i := 1;
while v[i] <> x do
  i := i+1;
od;
return i;
end;

s2 := function(c)
# generates points in 2-sphere of c-regular tree
local v, k, i, j;
v := [ ];
k := 1;
for i in [1..c] do
  for j in [1..c] do
    if i+j <> c+1 then # exclude reducible paths
      v[k] := [i,j];
```

```
    k := k+1;
   fi;
  od;
od;
return v;
end;

vPerm2i := function(i, cA, cB, A, B)
# generates i-th permutation in P2v
local w, j;
w := [ ];
for j in [1..cB*(cB-1)] do
  w[j] := indx(s2(cB), [B[cA+1-i][s2(cB)[j][1]],
          B[A[cA+1-i][s2(cB)[j][1]]][s2(cB)[j][2]]]);
od;
return PermList(w);
end;

P2v := function(A, B)
# generates P2v
local i, p, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
p := [ ];
for i in [1..cA/2] do
  p[i] := vPerm2i(i, cA, cB, A, B);
od;
return Group(p,());
end;

hPerm2j := function(j, cA, cB, A, B)
# generates j-th permutation in P2h
local w, i;
w := [ ];
for i in [1..cA*(cA-1)] do
  w[i] := indx(s2(cA), [cA+1-A[cA+1-s2(cA)[i][1]][j],
          cA+1-A[cA+1-s2(cA)[i][2]][B[cA+1-s2(cA)[i][1]][j]]]);
od;
return PermList(w);
end;

P2h := function(A, B)
# generates P2h
local j, p, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
p := [ ];
for j in [1..cB/2] do
  p[j] := hPerm2j(j, cA, cB, A, B);
```

```
od;
return Group(p,());
end;

Kh6 := function(A, B)
# generates K_h for m = 3
return Stabilizer(Stabilizer(Stabilizer(Stabilizer(Stabilizer(Stabilizer(P2h(A, B),
         [1, 2, 3, 4, 5], OnTuples),
         [6, 7, 8, 9, 10], OnSets),
         [11, 12, 13, 14, 15], OnSets),
         [16, 17, 18, 19, 20], OnSets),
         [21, 22, 23, 24, 25], OnSets),
         [26, 27, 28, 29, 30], OnSets);
end;
```

D.5 Computing a presentation of Γ

A finite presentation for Γ is obtained as follows (illustrated for $m = n = 3$):

```
F := FreeGroup("a1", "a2", "a3", "b1", "b2", "b3");
# free group generated by a_1, a_2, a_3, b_1, b_2, b_3
a1 := F.1;
a2 := F.2;
a3 := F.3;
b1 := F.4;
b2 := F.5;
b3 := F.6;

NL6a := function(n)
# map {1,...,2m} → E_h
local v;
if n=1 then v := a1;
  elif n=2 then v := a2;
    elif n=3 then v := a3;
    elif n=4 then v := a3^-1;
  elif n=5 then v := a2^-1;
elif n=6 then v := a1^-1;
fi;
return v;
end;

NL6b := function(n)
# map {1,...,2n} → E_v
local v;
if n=1 then v := b1;
  elif n=2 then v := b2;
    elif n=3 then v := b3;
    elif n=4 then v := b3^-1;
  elif n=5 then v := b2^-1;
```

```
elif n=6 then v := b1^-1;
fi;
return v;
end;

relation6 := function(A, B)
# generates mn relators of Γ
local i, j, rel, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
rel := [];
for i in [1..cA/2] do
  for j in [1..cB] do
    if not NL6a(i)*NL6b(j)*NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j]) in rel and
       not NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j])*NL6a(i)*NL6b(j) in rel and
       not NL6a(cA+1-A[i][j])^-1*NL6b(j)^-1*NL6a(i)^-1*NL6b(cB+1-B[i][j])^-1
       in rel then
       Add(rel,NL6a(i)*NL6b(j)*NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j]));
    fi;
  od;
od;
return rel;
end;

G := F / relation6(A,B); # definition of Γ

# e.g. AbelianInvariants(G); computes now Γ^ab
# LowIndexSubgroupsFpGroup(G, TrivialSubgroup(G), 8);
# computes all subgroups of low index (here of index ≤ 8),
# only reasonable for small index
```

D.6 A normal form program

Useful for other investigations are programs which bring a word in Γ in ab- and ba-normal form, see Proposition 6 (again illustrated for $m = n = 3$):

```
# F, a1, a2, a3, b1, b2, b3, NL6a(), NL6b() as in Appendix D.5

LN6a := function(w)
# map E_h → {1,...,2m}, inverse of NL6a
local n;
if w=a1 then n := 1;
  elif w=a2 then n := 2;
    elif w=a3 then n := 3;
    elif w=a3^-1 then n := 4;
  elif w=a2^-1 then n := 5;
elif w=a1^-1 then n := 6;
fi;
return n;
```

```
end;

LN6b := function(w)
# map E_v → {1,...,2n}, inverse of NL6b
local n;
if w=b1 then n := 1;
  elif w=b2 then n := 2;
    elif w=b3 then n := 3;
    elif w=b3^-1 then n := 4;
  elif w=b2^-1 then n := 5;
elif w=b1^-1 then n := 6;
fi;
return n;
end;

SetA6 := [a1, a2, a3, a3^-1, a2^-1, a1^-1];

SetB6 := [b1, b2, b3, b3^-1, b2^-1, b1^-1];

nfab := function(A,B,w)
# brings word w in ab-normal form
local i;
for i in [1..Length(w)-1] do
  if Subword(w,i,i) in SetB6 and Subword(w,i+1,i+1) in SetA6 then
    return nfab(A,B,SubstitutedWord(w,i,i+1,
         (NL6b(B[LN6a(Subword(w,i+1,i+1)^-1)][LN6b(Subword(w,i,i)^-1)])*
          NL6a(A[LN6a(Subword(w,i+1,i+1)^-1)][LN6b(Subword(w,i,i)^-1)]))^-1));
  fi;
od;
return w;
end;

nfba := function(A,B,w)
# brings word w in ba-normal form
local i;
for i in [1..Length(w)-1] do
  if Subword(w,i,i) in SetA6 and Subword(w,i+1,i+1) in SetB6 then
    return nfba(A,B,SubstitutedWord(w,i,i+1,
         NL6b(B[LN6a(Subword(w,i,i))][LN6b(Subword(w,i+1,i+1))])*
         NL6a(A[LN6a(Subword(w,i,i))][LN6b(Subword(w,i+1,i+1))])));
  fi;
od;
return w;
end;
```

D.7 Computing $\mathrm{Aut}(X)$

The following program generates all elements of $\mathrm{Aut}(X)$, where X is given by A and B (again illustrated for $m = n = 3$).

```
# F, a1, a2, a3, b1, b2, b3, NL6a(), NL6b() as in Appendix D.5

relation := function(A, B)
local i, j, k, rel, rel2, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
rel := [];
rel2 := [];
for i in [1..cA] do
  for j in [1..cB] do
    rel[cB*(i-1)+j] := NL6a(i)*NL6b(j)*NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j]);
  od;
od;
for k in [1..cA*cB] do
  rel2[k] := Subword(rel[k],2,4)*Subword(rel[k],1,1);
od;
return Union(rel,rel2);;
end;

LN := function(w,k1,k2,k3,k4,k5,k6,c)
local n;
if w=a1 then n := k1;
  elif w=a2 then n := k2;
    elif w=a3 then n := k3;
      elif w=b1 then n := k4;
         elif w=b2 then n := k5;
           elif w=b3 then n := k6;
           elif w=b3^-1 then n := c-k6;
         elif w=b2^-1 then n := c-k5;
      elif w=b1^-1 then n := c-k4;
    elif w=a3^-1 then n := c-k3;
  elif w=a2^-1 then n := c-k2;
elif w=a1^-1 then n := c-k1;
fi;
return n;
end;

NL := function(z)
local n;
if z=1 then n := a1;
  elif z=2 then n := a2;
    elif z=3 then n := a3;
      elif z=4 then n := b1;
         elif z=5 then n := b2;
           elif z=6 then n := b3;
           elif z=7 then n := b3^-1;
         elif z=8 then n := b2^-1;
      elif z=9 then n := b1^-1;
    elif z=10 then n := a3^-1;
```

```
  elif z=11 then n := a2^-1;
elif z=12 then n := a1^-1;
fi;
return n;
end;

permute := function(A,B)
local i1, i2, i3, j1, j2, j3, k, PL, L, cA, cB, c;
PL := [];
L := relation(A,B);
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
c := cA + cB;
for i1 in [1..c] do
  for i2 in Difference([1..c], [i1, c+1-i1]) do
    for i3 in Difference([1..c], [i1, c+1-i1, i2, c+1-i2])do
      for j1 in Difference([1..c], [i1, c+1-i1, i2, c+1-i2, i3, c+1-i3])do
        for j2 in Difference([1..c], [i1, c+1-i1, i2, c+1-i2, i3, c+1-i3, j1, c+1-j1])do
          for j3 in Difference([1..c],
                        [i1, c+1-i1, i2, c+1-i2, i3, c+1-i3, j1, c+1-j1, j2, c+1-j2])do
            for k in [1..Size(L)] do
              PL[k] := NL(LN(Subword(L[k],1,1),i1,i2,i3,j1,j2,j3,c+1))
                    * NL(LN(Subword(L[k],2,2),i1,i2,i3,j1,j2,j3,c+1))
                    * NL(LN(Subword(L[k],3,3),i1,i2,i3,j1,j2,j3,c+1))
                    * NL(LN(Subword(L[k],4,4),i1,i2,i3,j1,j2,j3,c+1));
            od;
            if Set(PL) = Set(L) then
              Print(NL(i1)," ",NL(i2)," ",NL(i3)," ",
                    NL(j1)," ",NL(j2)," ",NL(j3)," ","\n");
            fi;
          od;
        od;
      od;
    od;
  od;
od;
end;

permute(A,B);
```

For X as in Example 1, i.e. for

$$A = \begin{pmatrix} 1 & 1 & 5 & 3 & 1 & 1 \\ 3 & 3 & 3 & 4 & 6 & 3 \\ 2 & 5 & 1 & 2 & 2 & 2 \\ 5 & 6 & 2 & 5 & 5 & 5 \\ 4 & 4 & 4 & 1 & 3 & 4 \\ 6 & 2 & 6 & 6 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \\ 6 & 3 & 2 & 1 & 4 & 5 \\ 4 & 3 & 2 & 5 & 6 & 1 \\ 6 & 1 & 2 & 5 & 4 & 3 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix}$$

we get

```
permute(A,B);
a1 a2 a3 b1 b2 b3
a1^-1 a2^-1 a3^-1 b1^-1 b3 b2
```

D.8 Quaternion lattice program

We illustrate the construction of $\Gamma_{p,l}$ of Section 5 for the example $p = 3$, $l = 5$ (Example 52).

```
psi := function(v,x0,x1,x2,x3)
  return[[x0 + v*x1*E(4), v*x2 + v*x3*E(4)],[-v*x2 + v*x3*E(4), x0 - v*x1*E(4)]];
end;

a := [];
b := [];

a[1] := psi(1,1,0,1,1);
# $\psi(1+j+k)$
a[2] := psi(1,1,0,1,-1);
# $\psi(1+j-k)$
a[3] := psi(-1,1,0,1,-1);
# $\psi(1-j+k)$
a[4] := psi(-1,1,0,1,1);
# $\psi(1-j-k)$

b[1] := psi(1,1,2,0,0);
b[2] := psi(1,1,0,2,0);
b[3] := psi(1,1,0,0,2);
b[4] := psi(-1,1,0,0,2);
b[5] := psi(-1,1,0,2,0);
b[6] := psi(-1,1,2,0,0);

qAB := function(p,l)
local i, j, k, m, A, B;
A := NullMat(p+1,l+1);
B := NullMat(p+1,l+1);
for i in [1..p+1] do
  for j in [1..l+1] do
    for k in [1..l+1] do
      for m in [1..p+1] do
         if a[i]*b[j] = b[k]*a[m] or a[i]*b[j] = -b[k]*a[m] then
           A[i][j] := m;
           B[i][j] := k;
         fi;
      od;
    od;
  od;
od;
return([A,B]);
end;
```

```
A := qAB(3,5)[1];
B := qAB(3,5)[2];
```

gives

$$A = \begin{pmatrix} 3 & 3 & 2 & 4 & 4 & 2 \\ 1 & 4 & 3 & 1 & 3 & 4 \\ 4 & 2 & 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 3 & 2 & 3 \end{pmatrix},$$

$$B = \begin{pmatrix} 5 & 1 & 6 & 2 & 3 & 4 \\ 3 & 6 & 2 & 1 & 4 & 5 \\ 4 & 3 & 1 & 5 & 6 & 2 \\ 2 & 4 & 5 & 6 & 1 & 3 \end{pmatrix}.$$

E Miscellanea

E.1 Topology of $\mathrm{Aut}(\mathcal{T}_\ell)$

Throughout this section, let $\mathcal{T}_\ell$ be the ℓ-regular tree and $G = \mathrm{Aut}(\mathcal{T}_\ell)$ its group of automorphisms. We denote by X the countable vertex set of $\mathcal{T}_\ell$ endowed with the discrete topology. Let $X = \{x_1, x_2, \ldots\}$ be a fixed enumeration of X. For subsets $V, W \subseteq X$ and elements $x, v, w \in X$, we define $G_{V,W} := \{g \in G : g(V) \subseteq W\}$, the vertex stabilizer $G_x := G_{\{x\},\{x\}}$, the pointwise stabilizer $G_V := \cap_{x \in V} G_x$ and to simplify the notation we write $G_{v,W} := G_{\{v\},W}$, $G_{v,w} := G_{\{v\},\{w\}}$. We take the product topology on $\prod_{x\in X} X \cong X^X = \{f : X \to X\}$ and let $\mathcal{O}$ be the relative topology for $G \subset X^X$. Let $\pi_i : \prod_{x\in X} X \to X$ be the i-th projection. The product topology guarantees that these maps are continuous. Again, by definition of the product topology, a subbase for $\mathcal{O}$ is given by the sets $G_{v,W}$, where $v \in V \subseteq X$ and $W \subseteq X$. Since $G_{v,W} = \cup_{w\in W} G_{v,w}$, the family of sets $G_{v,w}$, where $v, w \in X$, is another subbase for $\mathcal{O}$. This topology $\mathcal{O}$ is sometimes called *topology of pointwise convergence* (or *topology of simple convergence*), since a sequence $(g_n)_{n\in\mathbb{N}}$ in G converges to $g \in G$ if and only if $(g_n(x))$ converges to $g(x)$ in X for all $x \in X$. Since X carries the discrete topology, this means that for each $x \in X$, there is an integer m such that $g_n(x) = g(x)$ for $n \geq m$. Note that $\mathcal{O}$ is the *compact open topology*, since this has as subbase the sets $G_{V,W}$, where $V \subset X$ is finite, $W \subseteq X$, and since

$$G_{V,W} = \bigcap_{i=1}^{n} \bigcup_{w\in W} G_{v_i,w},$$

where $V = \{v_1, \ldots, v_n\}$.

Proposition 78. $(G, \mathcal{O})$ *is a locally compact, totally disconnected, second countable, metrizable Hausdorff space. Moreover, it is a topological group, where we take the usual composition of elements in the group G.*

Proof. **Hausdorff:** X^X is Hausdorff as a product of Hausdorff spaces (see [38, Theorem III.5]), hence also its subspace G is Hausdorff.

Countable base: This follows immediately since X is countable and $\{G_{v,w} : v, w \in X\}$ is a subbase for $\mathcal{O}$.

Metrizable: Let ρ be the discrete metric on X, i.e. $\rho(v, w) := 0$ if $v = w$ and $\rho(v, w) := 1$ if $v \neq w$. We define for $g, h \in G$

$$d(g, h) := \sum_{i=1}^{\infty} \rho(g(x_i), h(x_i)).$$

Then d is a metric on G which induces $\mathcal{O}$ (see [17, Theorem 6.20]).

Locally compact: Let $v, w \in X$. If we can show that $G_{v,w}$ is compact, then any $g \in G$ has a compact neighbourhood. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $G_{v,w}$. By the local finiteness of $\mathcal{T}_\ell$, the set $\{g_n(x_i) : n \in \mathbb{N}\}$ is finite for each $i \in \mathbb{N}$. Therefore, there is an infinite subset $N_1 \subseteq \mathbb{N}$ such that the vertices $g_{n_1}(x_1)$ coincide for all $n_1 \in N_1$. Denote this common vertex by $g(x_1)$. Next, choose an infinite subset $N_2 \subseteq N_1$, such that $g_{n_2}(x_2)$ coincide for all $n_2 \in N_2$ and define $g(x_2) := g_{n_2}(x_2)$ $(n_2 \in N_2)$. Continuing this process $(i = 3, 4, \ldots)$ defines an element $g \in G_{v,w}$. By construction, g is a cluster point of $(g_n)_{n\in\mathbb{N}}$. This shows that $G_{v,w}$ is countably compact. But in a metric space, the notions of countably compactness and compactness are equivalent.

Note that G_x is a profinite group (see [20, Proposition 1.3.5]). Recall that a topological group is profinite if and only if it is compact and totally disconnected.

Moreover, note that X^X is *not* locally compact (this follows from [38, Theorem V.19]).

Separable: A metric space is separable if and only if it has a countable base (see [17, Corollary 7.21]).

Totally disconnected: We show that X^X is totally disconnected. Assume that $K \subset X^X$ is a connected subset such that $k_1, k_2 \in K$. Since the projections π_i are continuous, each image $\pi_i(K)$ is connected in X, i.e. a point. Thus $\pi_i(k_1) = \pi_i(k_2)$ for each i and therefore $k_1 = k_2$. G is totally disconnected as a subspace of X^X.

Topological group: Let $\mathcal{U}$ be the family of sets G_V, where V runs over *finite* subsets of X. Note that $G_V = \cap_{v \in V} G_{v,v}$ is open in G. We first show that $\mathcal{B}_1 := \{gU : g \in G, U \in \mathcal{U}\}$ is a base for some topology $\tilde{\mathcal{O}}$ on G such that $(G, \tilde{\mathcal{O}})$ (with the usual composition in the group G) is a topological group and then show that $\tilde{\mathcal{O}} = \mathcal{O}$.
The *sub*base $\mathcal{B}_1 = \{gU : g \in G, U \in \mathcal{U}\}$ generates a topology $\tilde{\mathcal{O}}$ on G, in particular, the family $\mathcal{B}_2$ of finite intersections of elements in $\mathcal{B}_1$ is a base for $\tilde{\mathcal{O}}$. Obviously, we have $\mathcal{B}_1 \subseteq \mathcal{B}_2$. If we can prove $\mathcal{B}_2 \subseteq \mathcal{B}_1$, then $\mathcal{B}_1$ is a base for $\tilde{\mathcal{O}}$ as claimed. Let

$$B_2 = \bigcap_{i=1}^{n} g_i U_i \quad (g_i \in G, U_i \in \mathcal{U})$$

be any element in $\mathcal{B}_2$ and let $h \in B_2$. Then $g_i^{-1} h \in U_i$ for each $i = 1, \dots, n$ and therefore $g_i^{-1} h U_i = U_i$ for each $i = 1, \dots, n$, using that $U_i = G_{V_i}$ for some finite $V_i \subset X$. Thus,

$$B_2 = \bigcap_{i=1}^{n} h U_i = h \left(\bigcap_{i=1}^{n} U_i \right) \in \mathcal{B}_1,$$

since $\cap_{i=1}^{n} U_i \in \mathcal{U}$. Recall that the map

$$\begin{aligned} \phi : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$

is continuous if for each $(g_1, g_2) \in G \times G$ and each open neighbourhood $\hat{U}$ of $g_1 g_2$ in G there is an open neighbourhood $\hat{V}$ of (g_1, g_2) in $G \times G$ such that $\phi(\hat{V}) \subset \hat{U}$.
So let $(g_1, g_2) \in G \times G$ and let $\hat{U} = \cup h_\iota U_\iota$ $(h_\iota \in G, U_\iota \in \mathcal{U})$ be an open neighbourhood of $g_1 g_2$ in G, say $g_1 g_2 = h_j u_j \in h_j U_j \subset \hat{U}$ with $U_j = G_{V_j}$. Then $g_2^{-1} G_{g_2(V_j)} g_2 U_j \subset U_j$. It follows that

$$\left(g_1 G_{g_2(V_j)}\right) (g_2 U_j) \subset g_1 g_2 U_j = h_j u_j U_j = h_j U_j \subset \hat{U}.$$

Since $g_1 G_{g_2(V_j)} \times g_2 U_j$ is an open neighbourhood of (g_1, g_2) in $G \times G$, we conclude that ϕ is continuous.
The proof of the continuity of the map $G \to G, g \mapsto g^{-1}$ is similar. We have to show that for each $g \in G$ and each open neighbourhood $\hat{U}$ of g^{-1} there is an open neighbourhood $\hat{V}$ of g such that $\hat{V}^{-1} \subset U$:
Let $g \in G$ and let $\hat{U} = \cup h_\iota U_\iota$ $(h_\iota \in G, U_\iota \in \mathcal{U})$ be an open neighbourhood of g^{-1}, say $g^{-1} = h_j u_j \in h_j U_j \subset \hat{U}$ with $U_j = G_{V_j}$ and define $\hat{V} = G_{g^{-1}(V_j)} \in \mathcal{U}$. Then $g \hat{V}^{-1} g^{-1} \subset U_j$ and

$$\left(g\hat{V}\right)^{-1} \subset g^{-1} U_j = h_j u_j U_j = h_j U_j \subset \hat{U}.$$

Since $g\hat{V}$ is an open neighbourhood of g, the map $g \mapsto g^{-1}$ is continuous and $(G, \tilde{\mathcal{O}})$ is a topological group.
We know that $\{G_{v,w} : v, w \in X\}$ is a subbase for $\mathcal{O}$ and $\{gU : g \in G, U = G_V, V \subset X \text{ finite}\}$ is a subbase for $\tilde{\mathcal{O}}$. In fact, $\mathcal{O} = \tilde{\mathcal{O}}$, because on one hand $G_{v,w} = gG_v$ for any $g \in G$ such that $g(v) = w$, and on the other hand

$$gG_V = \bigcap_{v \in V} G_{v,g(v)}.$$

$\square$

Proposition 79. *Let Γ be a subgroup of G and define $\Gamma_x := \Gamma \cap G_x$. Then the following three statements are equivalent:*

i) Γ *is discrete.*

ii) Γ_x *is finite for all* $x \in X$.

iii) Γ_x *is finite for some* $x \in X$.

Proof. i) $\Rightarrow$ ii): A discrete subgroup H of a Hausdorff topological group G is closed in G (see [32, Theorem 5.10]). Applying this theorem, Γ is closed in G and $\Gamma_x = \Gamma \cap G_x$ is closed in G_x, hence compact (since G_x is compact). But Γ_x is also discrete (being a subgroup of Γ), thus finite.

ii) $\Rightarrow$ iii): This is obvious.

iii) $\Rightarrow$ i): Write $\Gamma_x = \{\gamma_1, \ldots, \gamma_n\}$. For any $\gamma_i \in \Gamma_x \setminus \{1\}$ there is some (large) integer m_i such that $\gamma_i \notin \Gamma \cap G_{S(x,m_i)}$. Let m be the maximum of the m_i's, then $\Gamma \cap G_{S(x,m)} = \{1\}$. Since $G_{S(x,m)}$ is open in G, $\{1\}$ is open in Γ, and Γ is discrete ($\{\gamma\} = \{\gamma\}\{1\}$ is open in Γ). $\square$

Remark. By Proposition 79, G is not discrete, in particular $\{g\}$ is not open in G. However, $\{g\}$ is closed in G, since

$$\{g\} = G \setminus \bigcup_{i \in \mathbb{N}} G_{x_i, X \setminus \{g(x_i)\}}.$$

E.2 Growth of $(2m, 2n)$-groups

Let Γ be a finitely generated group and S a finite set generating Γ. Following [31], we define the *word length* $\ell_S(\gamma)$ of an element $\gamma \in \Gamma \setminus \{1\}$:

$$\ell_S(\gamma) := \min\{i : \gamma = s_1 \dots s_i,\ s_1, \dots, s_i \in S \cup S^{-1}\}, \quad (\ell_S(1) := 0),$$

for $k \in \mathbb{N}_0$ the *growth function*

$$k \mapsto \beta(\Gamma, S; k) := \#\{\gamma \in \Gamma : \ell_S(\gamma) \leq k\},$$

the corresponding *growth series*

$$B(\Gamma, S; z) := \sum_{k=0}^{\infty} \beta(\Gamma, S; k) z^k,$$

the *spherical growth function*

$$k \mapsto \sigma(\Gamma, S; k) := \beta(\Gamma, S; k) - \beta(\Gamma, S; k-1), \quad (\sigma(\Gamma, S; 0) := 1),$$

and the corresponding *spherical growth series*

$$\Sigma(\Gamma, S; z) := \sum_{k=0}^{\infty} \sigma(\Gamma, S; k) z^k = (1-z) B(\Gamma, S; z).$$

Proposition 80. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R(m,n) \rangle$ be a $(2m, 2n)$-group and take the standard generators $S := \{a_1, \dots, a_m, b_1, \dots, b_n\}$ of Γ.*

(1) The Cayley graph of (Γ, S) can be identified with the 1-skeleton of $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$, in particular the growth functions of (Γ, S) only depend on m and n.

(2)

$$\Sigma(\Gamma, S; z) = \frac{\left(\frac{1+z}{1-z}\right)^2}{\left(m - (m-1)\frac{1+z}{1-z}\right)\left(n - (n-1)\frac{1+z}{1-z}\right)} = \frac{1+z}{1-(2m-1)z} \cdot \frac{1+z}{1-(2n-1)z}$$

$$= 1 + (2m+2n)z + \left(2m(2m+2n-1) + 2n(2n-1)\right) z^2 + O(z^3)$$

(3) If $(m,n) \neq (1,1)$, then Γ is of exponential growth.

(4) If $m, n \geq 2$, then Γ is quasi-isometric to $F_2 \times F_2$.

Proof. (1) See [8, I.8A.2] for an explicit identification. Note that $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$ is the universal cover of the "Cayley complex", which is exactly our $(2m, 2n)$-complex X.

(2) By (1), $\Sigma(\Gamma, S; z) = \Sigma(F_m \times F_n, S; z)$. Note that

$$\Sigma(\mathbb{Z}, \{1\}; z) = \frac{1+z}{1-z}.$$

The claim follows now from the behaviour of the spherical growth series with respect to taking free and direct products, (see [31, Proposition VI.A 4]). As an intermediate step, we have for example

$$\Sigma(F_m, \{a_1, \dots, a_m\}; z) = \frac{1+z}{1-(2m-1)z}.$$

(3) This follows from the obvious fact that $F_m \times F_n$ contains a non-abelian free subgroup (namely $F_m \times \{1\}$ if $m \geq 2$ or $\{1\} \times F_n$ if $n \geq 2$).

(4) $F_m \times F_n$ is isomorphic to a finite index subgroup of $F_2 \times F_2$ (the index is $(m-1)(n-1)$), hence the groups are quasi-isometric by (1). (Note that for $\ell, \ell' \geq 3$, the tree $\mathcal{T}_\ell$ is quasi-isometric to $\mathcal{T}_{\ell'}$, see [8, Exercise I.8.20(2)]. This is a more general result than (4), since ℓ, ℓ' are allowed to be odd.)

$\square$

Example. Let Γ be a $(6,6)$-group. Then

$$\Sigma(\Gamma, S; z) = 1 + 12z + 96z^2 + 660z^3 + 4200z^4 + 25500z^5 + O(z^6)$$

and

$$B(\Gamma, S; z) = 1 + 13z + 109z^2 + 769z^3 + 4969z^4 + 30469z^5 + O(z^6).$$

E.3 Deficiency of $(2m, 2n)$-groups

Let G be a finitely presented group. The deficiency of a finite presentation P of G is the number of generators minus the number of relations in P. The *deficiency* $\mathrm{def}(G)$ of G is the maximum of the deficiency of P taken over all possible finite presentations of G. It is well-known (see [26, Lemma 1.2]) that

$$\mathrm{def}(G) \leq \mathrm{rank} H_1(G;\mathbb{Z}) - d(H_2(G;\mathbb{Z})), \tag{21}$$

where $d(H_2(G;\mathbb{Z})$ denotes the minimal number of generators of the second homology group of G with integer coefficients. The group G is called *efficient* if equality holds in (21).

Proposition 81. *Let Γ be a $(2m, 2n)$-group. Then Γ is efficient and $\mathrm{def}(\Gamma) = m + n - mn$.*

Proof. Since Γ has the finite presentation $\langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n)\rangle$, we have

$$\mathrm{def}(\Gamma) \geq m + n - mn.$$

On the other hand

$$\begin{aligned}
\mathrm{def}(\Gamma) &\leq \mathrm{rank}(H_1(\Gamma;\mathbb{Z})) - d(H_2(\Gamma;\mathbb{Z})) \\
&= \mathrm{rank}(H_1(\Gamma;\mathbb{Z})) - \mathrm{rank}(H_2(\Gamma;\mathbb{Z})) \\
&= 1 - \chi(\Gamma) \\
&= m + n - mn.
\end{aligned}$$

The inequality is (21) and the equalities above are described in [39, Section 6], where $\chi(\Gamma)$ is the Euler characteristic of the complex X (or the alternating sums of the ranks of the homology groups of Γ, which is the same here). $\square$

Remark. The deficiency for a $(2m, 2n)$-group Γ is attained by its standard presentation

$$\langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R(m,n)\rangle$$

as well as by the natural presentations of their amalgams (if they exist, see Proposition 2) $F_n *_{F_{1-2m+2mn}} F_{1-m+mn}$ and $F_m *_{F_{1-2n+2mn}} F_{1-n+mn}$.

Remark. There are non-efficient torsion-free groups, see [45].

E.4 Certain regular graphs associated to a $(2m, 2n)$-group

Following an idea of Mozes ([51]), we associate to a $(2m, 2n)$-group Γ two infinite families of finite regular graphs $(X_k(\Gamma))_{k\in\mathbb{N}}$ and $(Y_k(\Gamma))_{k\in\mathbb{N}}$. The vertex set of $X_k(\Gamma)$ is $E_h^{(k)}$ and the vertex set of $Y_k(\Gamma)$ is $E_v^{(k)}$. Two vertices $a, \tilde{a} \in E_h^{(k)}$ are connected by a geometric edge if and only if $\rho_v(b)(a) = \tilde{a}$ for some $b \in E_v$. In this case, b and b^{-1} are oriented edges such that $o(b) = a$, $t(b) = \tilde{a}$, $\overline{b} = b^{-1}$. Similarly, two vertices $b, \tilde{b} \in E_v^{(k)}$ are connected by a geometric edge if and only if $\rho_v(a)(b) = \tilde{b}$ for some $a \in E_h$. We list some obvious properties of $X_k(\Gamma)$ (the properties of $Y_k(\Gamma)$ are analogous):

- $X_k(\Gamma)$ has $2m(2m-1)^{k-1}$ vertices.
- $X_k(\Gamma)$ is $2n$-regular.
- $X_k(\Gamma)$ is connected if and only if $P_h^{(k)}$ is transitive on $E_h^{(k)}$.
- $X_k(\Gamma)$ is connected for each k if and only if $\mathrm{pr}_1(\Gamma)$ is locally ∞-transitive.
- If $X_k(\Gamma)$ is not connected, then $X_l(\Gamma)$ is not connected for each $l \geq k$.
- If $X_k(\Gamma)$ has no loops, then $X_l(\Gamma)$ has no loops for each $l \geq k$.

Less obvious is

Proposition 82. *(Mozes [51]) If $\Gamma = \Gamma_{p,l}$ as in Section 5.2, then $(X_k(\Gamma))_{k\in\mathbb{N}}$ and $(Y_k(\Gamma))_{k\in\mathbb{N}}$ are* Ramanujan graphs, *i.e. for every $k \in \mathbb{N}$ and every eigenvalue λ of the adjacency matrix of $X_k(\Gamma)$, either $\lambda = \pm(l+1)$ or $|\lambda| \leq 2\sqrt{l}$, and for every eigenvalue λ of the adjacency matrix of $Y_k(\Gamma)$, either $\lambda = \pm(p+1)$ or $|\lambda| \leq 2\sqrt{p}$.*

Question 19. *Are there other $(2m, 2n)$-groups Γ such that $(X_k(\Gamma))_{k\in\mathbb{N}}$ are Ramanujan graphs?*

See Figure 20 and 21 for a visualization of $Y_1(\Gamma_{3,5})$ and $X_2(\Gamma_{3,5})$, where $\Gamma_{3,5}$ is the $(4,6)$-group of Section 5.4.3.

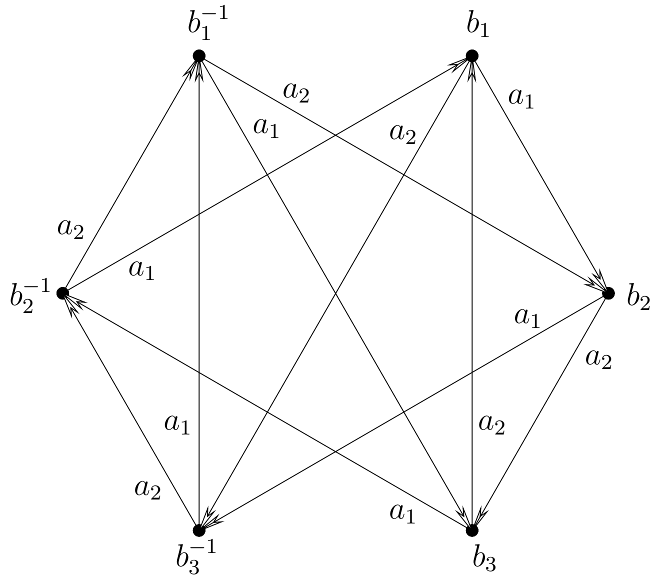


Figure 20: $Y_1(\Gamma_{3,5})$

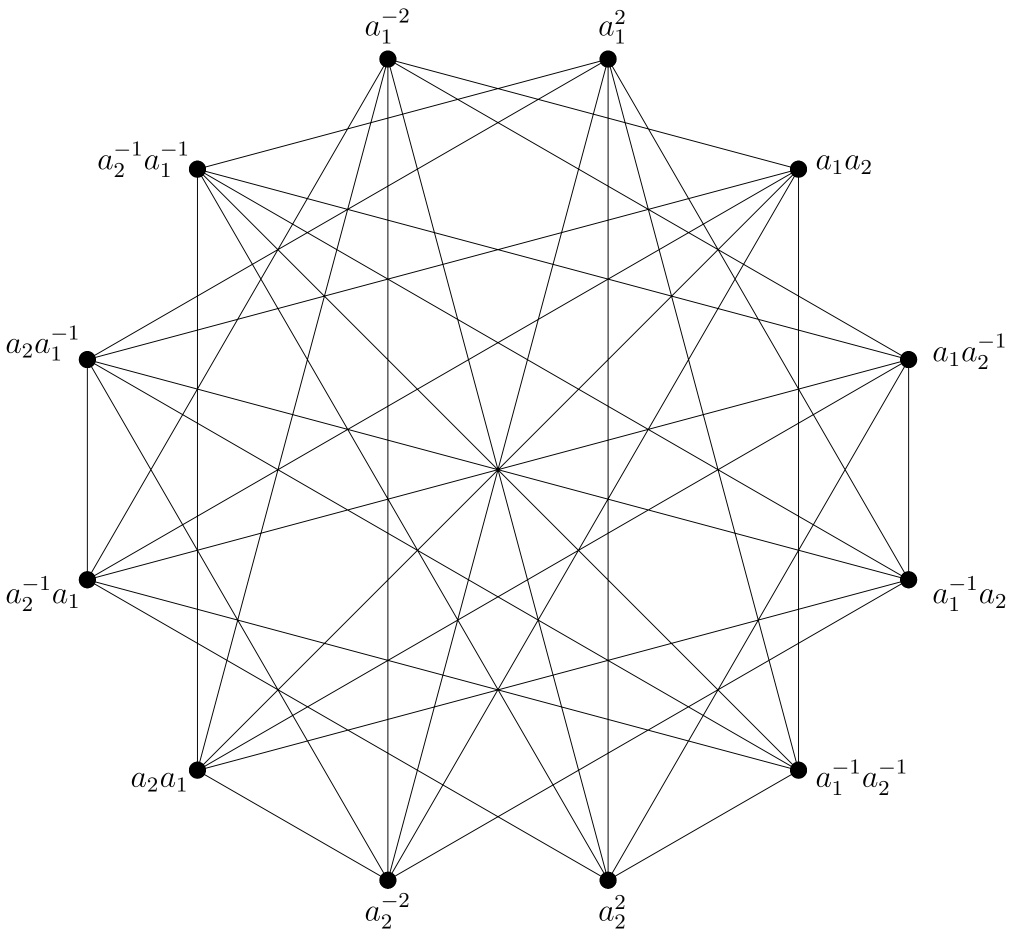


Figure 21: Geometric realization of $X_2(\Gamma_{3,5})$

E.5 History of simple groups and free amalgams

We give some selected history of finitely presented (generated) infinite simple groups and amalgams of finitely generated free groups. (In this section the term free group always means a non-abelian free group)

- **Aleksandr G. Kuroš 1944** ([40]) He asked for the existence of a finitely generated infinite simple group. (This was positively answered in [33])
- **Graham Higman 1951** ([33]) He gave the first existence proof of a finitely generated infinite simple group and asked for the existence of a finitely presented infinite simple group: "Can an infinite simple group have not only a finite set of generators, but also a finite set of defining relations?" (This was positively answered by Richard J. Thompson in 1965)
- **Ruth Camm 1953** ([18]) She constructed uncountably many finitely generated infinite simple groups of the form $F_2 *_{F_\infty} F_2$. These groups are torsion-free, 2-generated, but not finitely presentable (by [3]).
- **Richard J. Thompson 1965** (in unpublished notes) He defined two finitely presented infinite simple groups $\widehat{\mathcal{C}}$ (often called T) and $\widehat{V}$ (often called V). They are not torsion-free.

He defined also a third interesting group $\widehat{\mathbb{P}}$ (often called F) which is torsion-free but not simple. For an introduction to these three groups, see [19].

- **Peter M. Neumann 1973** ([56]) "At one time I had hoped that one might construct a finitely presented simple group as a generalised free product of two free groups A, B of finite rank amalgamating finitely generated subgroups H and K. Joan Landman-Dyer and I showed quite easily that if H has infinite index in A or K has infinite index in B then such a group G is not simple." For a proof that G is even SQ-universal under these conditions, see [63, Corollary 2]. For an alternative proof that G (again provided $[A : H]$ or $[B : K]$ is infinite) is not simple, see [36, Corollary 2]. Then Neumann posed the following problems (which appeared also in the Kourovka notebook): "Let $G = A *_{H=K} B$ where A, B are non-abelian free groups of finite rank and $|A : H|$, $|B : K|$ are finite. (a) Can it happen that G is simple? (b) Is G always SQ-universal?" ((a) was positively answered in [14]; consequently the answer to (b) is no)

- **Graham Higman 1974** ([34]) He generalized Thompson's group V to an infinite family of finitely presented infinite simple groups.

- **Dragomir Ž. Djoković 1981** ([25]) His finitely presented infinite "simple" group with bounded torsion turned out to be *not* simple.

- **Elisabeth A. Scott 1984** ([64]) She constructed another family of finitely presented infinite simple groups, related to the Higman groups.

- **Kenneth S. Brown 1985** ([10]) He generalized the Thompson groups T, V and established some finiteness properties.

- **Kenneth S. Brown 1989** ([11]) He showed that Thompson's group V can be written as a ("positively curved, realizable") triangle of groups with finite vertex groups S_5, S_6, S_7.

- **Meenaxi Bhattacharjee 1994** ([6]) She gave a construction of an amalgam $F_3 *_{F_{13}} F_3$ without non-trivial finite quotient. This group is "nearly simple" in her terminology, but it is not known whether it has proper infinite quotients, or it is simple. More examples like this appear in [6] and [7].

- **Geoffrey Mess** (in [57, Problem 5.11 (C)] **1995**) "Let X be a finite aspherical complex. Is there an example of an X with simple fundamental group?" (His question was positively answered in [14])

- **Daniel T. Wise 1996** ([69]) He constructed a square complex without a non-trivial finite covering and asked: "Does there exist a CSC with (non-trivial) simple π_1? I guess that one does exist." (where CSC stands for complete square complex, any $(2m, 2n)$-complex is CSC) (Again, this was positively answered in [14])

- **Marc Burger, Shahar Mozes 1997** ([14]) They constructed an infinite family of finitely presented torsion-free simple groups which are amalgams of finitely generated free groups and thereby solved many open problems mentioned above (Neumann, Mess, Wise).

- **Claas E. Röver 1999** ([62]) He gave a construction of finitely presented infinite simple groups that contain Grigorchuk groups.

References

[1] Ballmann, Werner, *Spaces of nonpositive curvature*, Jahresber. Deutsch. Math.-Verein. **103**(2001), no. 2, 52–65.

[2] Bass, Hyman, Lubotzky, Alexander, *Tree lattices*, With appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits. Progress in Mathematics, 176. Birkhäuser Boston, Inc., Boston, MA, 2001.

[3] Baumslag, Gilbert, *A remark on generalized free products*, Proc. Amer. Math. Soc. **13**(1962), 53–54.

[4] Baumslag, Gilbert, *Automorphism groups of residually finite groups*, J. London Math. Soc. **38**(1963), 117–118.

[5] Bestvina Mladen, *Questions in geometric group theory*, version of August 22, 2000, `http://www.math.utah.edu/~bestvina`

[6] Bhattacharjee, Meenaxi, *Constructing finitely presented infinite nearly simple groups*, Comm. Algebra **22**(1994), no. 11, 4561–4589.

[7] Bhattacharjee, Meenaxi, *Some interesting finitely presented infinite amalgamated free products*, Quart. J. Math. Oxford Ser. (2) **48**(1997), no. 189, 1–10.

[8] Bridson, Martin R.; Haefliger, André, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.

[9] Bridson, Martin R.; Wise, Daniel T., *$\mathcal{VH}$ complexes, towers and subgroups of $F \times F$*, Math. Proc. Cambridge Philos. Soc. **126**(1999), no. 3, 481–497.

[10] Brown, Kenneth S., *Finiteness properties of groups*, Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), J. Pure Appl. Algebra **44**(1987), no. 1-3, 45–75.

[11] Brown, Kenneth S., *The geometry of finitely presented infinite simple groups*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), 121–136, Math. Sci. Res. Inst. Publ., 23, Springer, New York, 1992.

[12] Buekenhout, Francis; Leemans, Dimitri, *On the list of finite primitive permutation groups of degree ≤ 50*, J. Symbolic Comput. **22**(1996), no. 2, 215–225.

[13] Burger, Marc; Monod Nicolas, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12**(2002), no. 2, 219–280.

[14] Burger, Marc; Mozes, Shahar, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sér. I Math. **324**(1997), no. 7, 747–752.

[15] Burger, Marc; Mozes, Shahar, *Groups acting on trees: from local to global structure*, Inst. Hautes Études Sci. Publ. Math. No. **92**(2000), 113–150(2001).

[16] Burger, Marc; Mozes, Shahar, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. No. **92**(2000), 151–194(2001).

[17] Cain, George L., *Introduction to general topology*, Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1994.

[18] Camm, Ruth, *Simple free products*, J. London Math. Soc. **28**(1953), 66–76.

[19] Cannon, James W.; Floyd, William J.; Parry, Walter R., *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) **42**(1996), no. 3-4, 215–256.

[20] Choucroun, Francis M., *Analyse harmonique des groupes d'automorphismes d'arbres de Bruhat-Tits* (French) [Harmonic analysis of the automorphism groups of Bruhat-Tits trees], Mém. Soc. Math. France (N.S.) No. **58**, (1994).

[21] Cohen, Daniel E., *A topological proof in group theory*, Proc. Cambridge Philos. Soc. **59**(1963), 277–282.

[22] Davidoff, Giuliana; Sarnak, Peter; Valette, Alain, *Elementary number theory, group theory, and Ramanujan graphs* exact ref

[23] Dickson, Leonard E., *Arithmetic of quaternions*, Proc. London Math. Soc. (2) **20**(1922), 225–232.

[24] Dixon, John D.; Mortimer, Brian, *Permutation groups*, Graduate Texts in Mathematics, 163. Springer-Verlag, New York, 1996.

[25] Djoković, Dragomir Ž., *Another example of a finitely presented infinite simple group*, J. Algebra **69**(1981), no. 2, 261–269.
A correction, a retraction, and addenda to this paper: J. Algebra **82**(1983), no. 1, 285–293.

[26] Epstein, David B. A., *Finite presentations of groups and* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) **12**(1961), 205–212.

[27] Fujiwara, Koji, *The second bounded cohomology of an amalgamated free product of groups*, Trans. Amer. Math. Soc. **352**(2000), no. 3, 1113–1129.

[28] GAP — Groups, Algorithms, and Programming, Version 4.2; 2000, `http://www.gap-system.org`

[29] Glasner, Yair, *Ramanujan graphs with small girth*, Combinatorica exact ref

[30] Marshall Hall Jr., *A topology for free groups and related groups*, Ann. of Math. (2) **52**(1950), 127–139.

[31] de la Harpe, Pierre, *Topics in geometric group theory*, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[32] Hewitt, Edwin; Ross, Kenneth A., *Abstract harmonic analysis*, Vol. I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 115. Springer-Verlag, Berlin-New York, 1979.

[33] Higman, Graham, *A finitely generated infinite simple group*, J. London Math. Soc. **26**(1951), 61–64.

[34] Higman, Graham, *Finitely presented infinite simple groups*, Notes on Pure Mathematics, No. 8 (1974). Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974.

[35] Ireland, Kenneth; Rosen, Michael, *A classical introduction to modern number theory*, Revised edition of Elements of number theory. Graduate Texts in Mathematics, 84. Springer-Verlag, New York-Berlin, 1982.

[36] Ivanov, Sergei V.; Schupp, Paul E., *A remark on finitely generated subgroups of free groups*, Algorithmic problems in groups and semigroups (Lincoln, NE, 1998), 139–142, Trends Math., Birkhäuser Boston, Boston, MA, 2000.

[37] Kari, Jarkko; Papasoglu, Panagiotis, *Deterministic aperiodic tile sets*, Geom. Funct. Anal. **9**(1999), no. 2, 353–369.

[38] Kelley, John L., *General topology*, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.]. Graduate Texts in Mathematics, No. 27. Springer-Verlag, New York-Berlin, 1975.

[39] Kimberley, Jason S., Robertson, Guyan, *Groups acting on products of trees, tiling systems and analytic K-theory*, New York J. Math. **8**(2002), 111–131 (electronic).

[40] Kuroš, Aleksandr G., *Teoriya Grupp* (Russian) [Theory of Groups] OGIZ, Moscow-Leningrad, 1944.

[41] Liu, Guoyang; Robertson, Lewis C., *Free subgroups of* $\mathrm{SO}_3(\mathbb{Q})$, Comm. Algebra **27**(1999), no. 4, 1555–1570.

[42] Long, Darren D.; Niblo, Graham A., *Subgroup separability and* 3*-manifold groups*, Math. Z. **207**(1991), no. 2, 209–215.

[43] Lubotzky, Alexander, *Discrete groups, expanding graphs and invariant measures*, With an appendix by Jonathan D. Rogawski. Progress in Mathematics, 125. Birkhäuser Verlag, Basel, 1994.

[44] Lubotzky, Alexander; Phillips, Ralph S.; Sarnak, Peter, *Ramanujan graphs*, Combinatorica **8**(1988), no. 3, 261–277.

[45] Lustig, Martin, *Non-efficient torsion-free groups exist*, Comm. Algebra **23**(1995), no. 1, 215–218.

[46] Lyndon, Roger C., *Dependence and independence in free groups*, J. Reine Angew. Math. **210**(1962), 148–174.

[47] Lyndon, Roger C.; Schupp, Paul E., *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York, 1977.

[48] MAGMA: Bosma, Wieb; Cannon, John; Playoust, Catherine, *The Magma algebra system. I. The user language*, Computational algebra and number theory (London, 1993). J. Symbolic Comput. **24**(1997), no. 3-4, 235–265.

[49] MAGNUS: A graphically-oriented software system for computational group theory, Version 3.2.1; 1999. (http://www.grouptheory.org)

[50] Malcev, Anatoliĭ I., *On the faithful representation of infinite groups by matrices*, Amer. Math. Soc. Transl. (2) **45**(1965), 1–18. Russian original: *On isomorphic matrix representations of infinite groups*, Rec. Math. [Mat. Sbornik] N.S. 8(50), (1940), 405–422.

[51] Mozes, Shahar, *A zero entropy, mixing of all orders tiling system*, Symbolic dynamics and its applications (New Haven, CT, 1991), 319–325, Contemp. Math., 135, Amer. Math. Soc., Providence, RI, 1992.

[52] Mozes, Shahar, *On closures of orbits and arithmetic of quaternions*, Israel J. Math. **86**(1994), no. 1-3, 195–209.

[53] Mozes, Shahar, *Actions of Cartan subgroups*, Israel J. Math. **90**(1995), no. 1-3, 253–294.

[54] Neumann, Bernhard H., *Some remarks on infinite groups*, J. London Math. Soc. **12**(1937), 120–127.

[55] Neumann, Bernhard H., *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London. Ser. A. **246**(1954), 503–554.

[56] Neumann, Peter M., *The SQ-universality of some finitely presented groups*, Collection of articles dedicated to the memory of Hanna Neumann, I., J. Austral. Math. Soc. **16**(1973), 1–6.

[57] *Problems in low-dimensional topology*, Edited by Rob Kirby. AMS/IP Stud. Adv. Math., 2.2, Geometric topology (Athens, GA, 1993), 35–473, Amer. Math. Soc., Providence, RI, 1997.

[58] Myasnikov, Alexei G.; Remeslennikov, Vladimir N., *Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups*, Internat. J. Algebra Comput. **6**(1996), no. 6, 687–711.

[59] quotpic: Holt, Derek F.; Rees, Sarah, *A graphics system for displaying finite quotients of finitely presented groups*, Groups and computation (New Brunswick, NJ, 1991), 113–126, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 11, Amer. Math. Soc., Providence, RI, 1993.

[60] Robertson, Guyan; Sinclair, Allan M.; Smith Roger R., *Strong singularity for subalgebras of finite factors*, Internat. J. Math. **14**(2003), no. 3, 235–258.

[61] Robinson, Raphael M., *Undecidability and nonperiodicity for tilings of the plane*, Invent. Math. **12**(1971), 177–209.

[62] Röver, Claas E., *Constructing finitely presented simple groups that contain Grigorchuk groups*, J. Algebra **220**(1999), no. 1, 284–313.

[63] Schupp, Paul E., *Cancellation theory over free products with amalgamation*, Math. Ann. **193**(1971), 255–264.

[64] Scott, Elizabeth A., *A construction which can be used to produce finitely presented infinite simple groups*, J. Algebra **90**(1984), no. 2, 294–322.

[65] Sela, Zlil, *Endomorphisms of hyperbolic groups. I. The Hopf property*, Topology **38**(1999), no. 2, 301–321.

[66] Serre, Jean-Pierre, *Trees*, Translated from the French by John Stillwell. Springer-Verlag, Berlin-New York, 1980.

[67] Wang, Hao, *Proving Theorems by Pattern Recognition. II.*, Bell System Tech. J. **40**(1961), 1–41.

[68] Wiegold, James; Wilson, John S., *Growth sequences of finitely generated groups*, Arch. Math. (Basel) **30**(1978), no. 4, 337–343.

[69] Wise, Daniel T., *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups*, PhD thesis, Princeton University, 1996.

Diego Rattaggi
ETH Zürich
D-MATH, HG D54
8092 Zürich
Switzerland

e-mail: `rattaggi@math.ethz.ch`